1ZA3 (SECTION C01) - ENGINEERING MATHEMATICS I

Lecture 27

Last time

SUBSTITUTION RULE ← Reverses Chain Rule

In $\int f(x)\,dx$, find a piece $u(x)$ with $\frac{du}{dx}$ a factor (up to a constant multiple) in $f(x)$.

Then use $\left(\frac{du}{dx}\right) = du/dx$ ← as though 2 numbers & substitute.

(Formally: if $u = g(x)$, $\int f(g(x)) \cdot g'(x)\,dx = \int f(u)\,du$.)

Definite Integral Version:

$$\int_a^b f(g(x)) \cdot g'(x)\,dx = \int_{g(a)}^{g(b)} f(u)\,du$$

Values over which x ranges

Corresponding values over which u ranges

Justification

As before if $F(x)$ is an antiderivative of $f(x)$,

then $\frac{d}{dx}\left(F(g(x))\right) = F'(g(x))\,g'(x) = f(g(x))\,g'(x)$

So by F.T.C. Part II

$$\int_a^b f(g(x))\,g'(x)\,dx = \Big[F(g(x))\Big]_a^b$$

$$= F(g(b)) - F(g(a)) = \Big[F(u)\Big]_{g(a)}^{g(b)}$$

Last time we had $\displaystyle = \int_{g(a)}^{g(b)} f(u)\,du.$

$$\int_0^1 x e^{x^2}\,dx = \left[\tfrac{1}{2}e^{x^2}\right]_0^1 = \tfrac{1}{2}e^1 - \tfrac{1}{2}e^0 = \tfrac{1}{2}(e-1)$$

We found $\left(\frac{1}{2}e^{x^2}\right)' = xe^{x^2}$

Or we could do:

$$\int_0^1 xe^{x^2}\,dx \underset{u=x^2}{=} \int_{0^2}^{1^2} \frac{1}{2}e^u\,du = \int_0^1 \frac{1}{2}e^u\,du = \left[\frac{1}{2}e^u\right]_0^1$$

$$= \frac{1}{2}e^1 - \frac{1}{2}e^0$$

$$= \frac{1}{2}(e-1).$$

<u>Example</u> Find $\displaystyle\int_1^2 \frac{\sin(\pi/x)}{x^2}\,dx.$

<u>Solution</u> Substitution $u = \dfrac{\pi}{x}$. Then $\dfrac{du}{dx} = -\dfrac{\pi}{x^2}$

$$\int_1^2 \frac{\sin(\pi/x)}{x^2}\,dx = \int_{\pi/1}^{\pi/2} \frac{\sin(u)}{\cancel{x^2}}\,\frac{du\,\cancel{x^2}}{-\pi} \Leftarrow \left\{ \rightsquigarrow dx = \frac{du}{(-\pi/x^2)} = \frac{du\cdot x^2}{-\pi} \right.$$

$$= -\frac{1}{\pi}\int_{\pi}^{\pi/2} \sin(u)\,du = \frac{1}{\pi}\int_{\pi/2}^{\pi} \sin(u)\,du$$

$$= \frac{1}{\pi}\Big[-\cos(u)\Big]_{\pi/2}^{\pi}$$

$$= \frac{1}{\pi}\left(-\cos(\pi) - \left(-\cos(\pi/2)\right)\right)$$

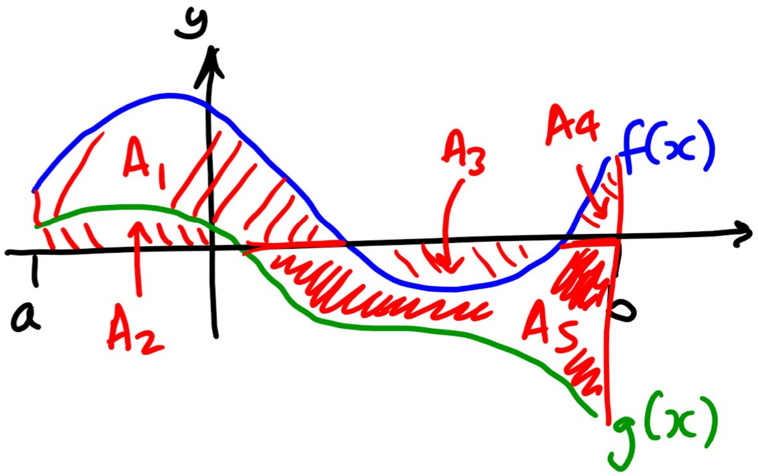
$$= \frac{1}{\pi}\left(-(-1) - (-0)\right) = \underline{\underline{\frac{1}{\pi}}}$$

UP TO HERE FOR TEST #2

APPLICATIONS OF INTEGRATION

6.1 Areas between curves

2 functions on $[a,b]$ $f(x) \geq g(x)$



Here $\displaystyle\int_a^b f(x)\,dx = A_1 + A_2 + A_4 - A_3$

Area above x-axis (below $y=f(x)$)

Area below x-axis above $y=f(x)$

$$\int_a^b g(x)\,dx = \underbrace{A_2}_{\text{area below } y=g(x) \text{ above } x\text{-axis}} - \underbrace{A_5 - A_3}_{A_5+A_3 = \text{area below } x\text{-axis above } y=g(x).}$$

Area between $f(x)$ & $g(x)$ $= A_1 + A_5 + A_4$

Disregard where the x-axis is!!!

$$= (A_1 + A_2 + A_4 - A_3)$$

$$- (A_2 - A_5 - A_3)$$

$$= \int_a^b f(x)\,dx - \int_a^b g(x)\,dx$$

In general, if $f(x) \geq g(x)$ on $[a,b]$

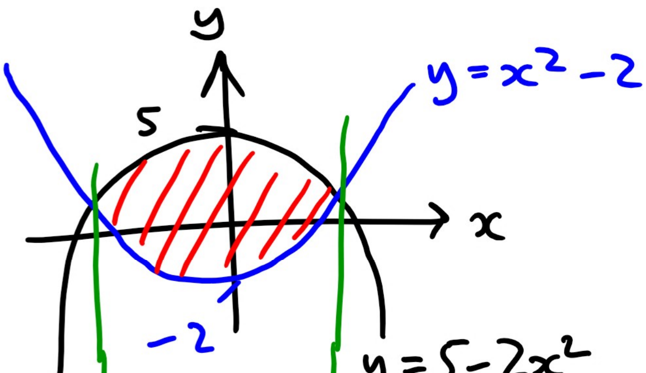
the area of the region bounded by $y = f(x)$

$y = g(x)$ is $\displaystyle\int_a^b \left(f(x) - g(x)\right) dx$.

Example

Find the area of the region enclosed by $y = 5 - 2x^2$ and $y = x^2 - 2$.

Solution



$a = -\sqrt{7/3}$ $\quad$ $+\sqrt{7/3} = b$

Points of intersection :

$x^2 - 2 = 5 - 2x^2 \rightarrow x = \pm\sqrt{\frac{7}{3}}$

$3x^2 = 7$

$x^2 = 7/3$

Need to identify what plays roles of $f(x)$, $g(x)$ and $[a,b]$.

So here $f(x) = 5 - 2x^2$

$g(x) = x^2 - 2$

Area $=$

$$\int_{-\sqrt{7/3}}^{\sqrt{7/3}} (5 - 2x^2) - (x^2 - 2)\, dx$$

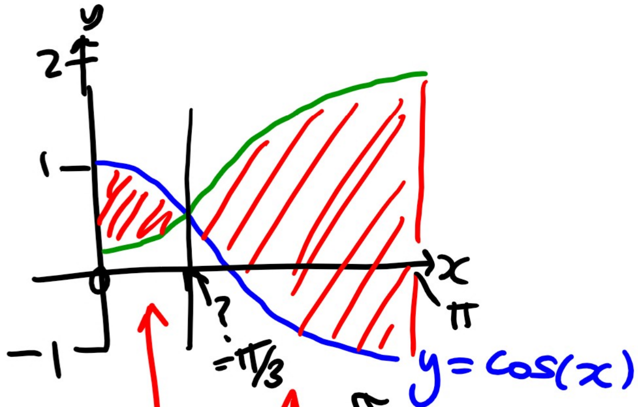
$$= \int_{-\sqrt{7/3}}^{\sqrt{7/3}} 7 - 3x^2\, dx$$

$$= \left[7x - x^3\right]_{-\sqrt{7/3}}^{\sqrt{7/3}}$$

$$= \left(7\left(\sqrt{\tfrac{7}{3}}\right) - \left(\sqrt{\tfrac{7}{3}}\right)^3\right) - \left(7\left(-\sqrt{\tfrac{7}{3}}\right) - \left(-\sqrt{\tfrac{7}{3}}\right)^3\right)$$

Example Find the area bounded by $y = \cos(x)$ and $y = 1 - \cos(x)$ on $[0, \pi]$.

Solution



Point of intersection

? given by

$\cos(x) = 1 - \cos(x)$

$\Rightarrow 2\cos(x) = 1$

$\Rightarrow \cos(x) = 1/2$

$\Rightarrow x = \pi/3$

2 regions

$f(x) = \cos(x) \geq 1 - \cos(x) = g(x)$ | $f(x) = 1 - \cos(x) \geq \cos(x) = g(x)$

$$\int_0^{\pi/3} \underbrace{\cos(x) - (1 - \cos(x))}_{2\cos(x) - 1 \,\geq\, 0 \text{ on } [0, \pi/3]} dx \quad + \quad \int_{\pi/3}^{\pi} \underbrace{(1 - \cos(x)) - \cos(x)}_{1 - 2\cos(x) \,\geq\, 0 \text{ on } [\pi/3, \pi]} dx$$

$= \#$ $\qquad$ $= \#$

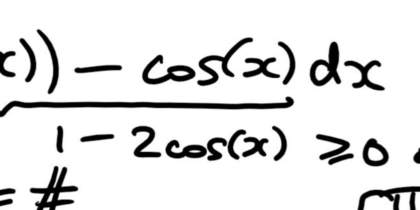
$= \left[2\sin(x) - x\right]_0^{\pi/3} = \left(2\sin(\pi/3) - \pi/3\right) - 0 = \sqrt{3} - \frac{\pi}{3}$

$= \left[x - 2\sin(x)\right]_{\pi/3}^{\pi} = (\pi - 0) - \left(\frac{\pi}{3} - \sqrt{3}\right) = \frac{2\pi}{3} + \sqrt{3}$

TOTAL $= \pi/3 + 2\sqrt{3}$

Notice: 2 integrands are almost same — both $= |2\cos(x) - 1|$

In general: if $\exists$ 2 functions $f(x)$, $g(x)$ & we have we want area between them, on $[a, b]$, this is given by:



$$\int_a^b |f(x) - g(x)|\, dx.$$

So this gives us a shortcut instead of having to break up the area into several regions.