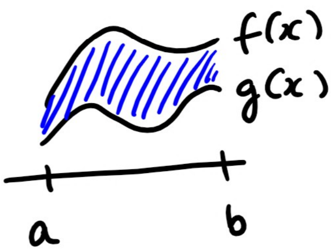
1ZA3 (SECTION C01) – ENGINEERING MATHEMATICS I

Lecture 28

Last time

AREAS BETWEEN CURVES

If $f(x) \geqslant g(x)$ on $[a,b]$, then the area of the region between $y=f(x)$ and $y=g(x)$ is given by



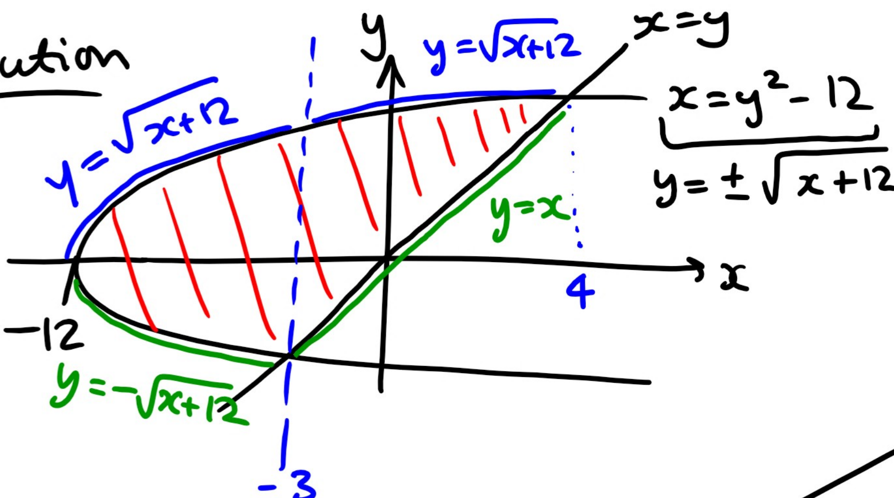
$$\int_a^b \underbrace{f(x) - g(x)}\,dx.$$

If $f(x)$ & $g(x)$ cross this is $|f(x)-g(x)|$.

Example Find the area of the region enclosed by

$$x = y^2 - 12 \quad \text{and} \quad x = y.$$

Solution



$$\underbrace{x = y^2 - 12}_{y = \pm\sqrt{x+12}}$$

2 options

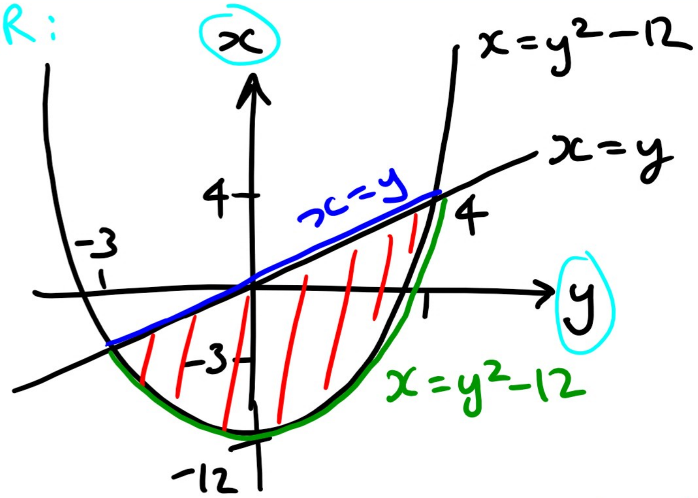
① Split into 2 regions of form

Left : $$\int_{-12}^{-3} \sqrt{x+12} - \left(-\sqrt{x+12}\right) dx$$

Right : $$\int_{-3}^{4} \sqrt{x+12} - x\,dx$$

So total area is the sum of these two nasty integrals.

OR:



x given as a function of y !

$$\text{Area} = \int_{-3}^{4} y - (y^2 - 12)\, dy$$

-3 ↑ on y-axis

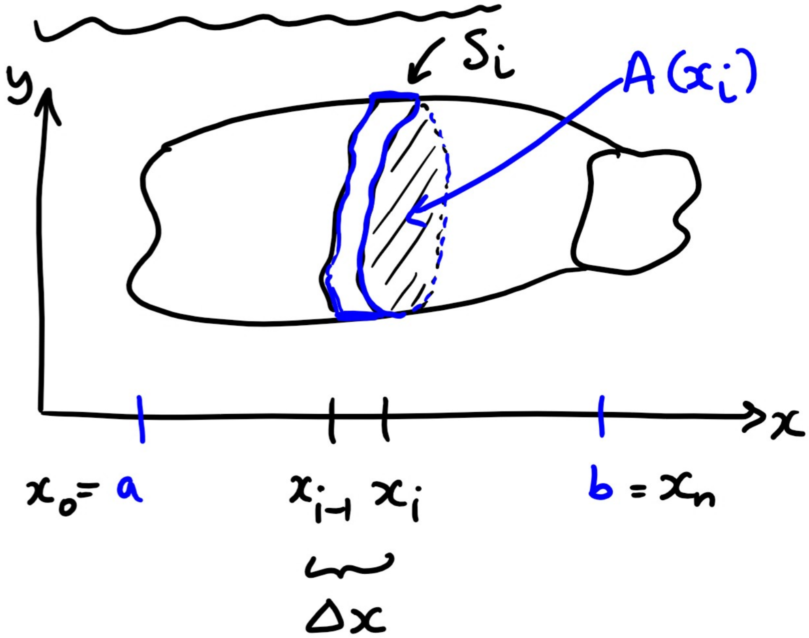
$$= \left[\frac{y^2}{2} - \frac{y^3}{3} + 12y \right]_{-3}^{4}$$

$$= \frac{4^2}{2} - \frac{4^3}{3} + 12(4) - \left(\frac{(-3)^2}{2} - \frac{(-3)^3}{3} + 12(-3) \right)$$

$$= \frac{343}{6}.$$

6.2 Volumes

3D analog to area problem



← To approximate the volume of this solid

- slice up the solid into n pieces vertically, called S_i ($i = 1, \dots, n$)

at evenly spaced x-values of width $\Delta x = \frac{b-a}{n}$

- Approximate volume of S_i (ith slice) by volume of straight-sided shape with width Δx & side area = $A(x_i)$ = cross-sectional area of solid at $x = x_i$.

i.e. Volume of $S_i \approx \Delta x . A(x_i)$

- So total volume $\approx \sum_{i=1}^{n} \Delta x . A(x_i)$

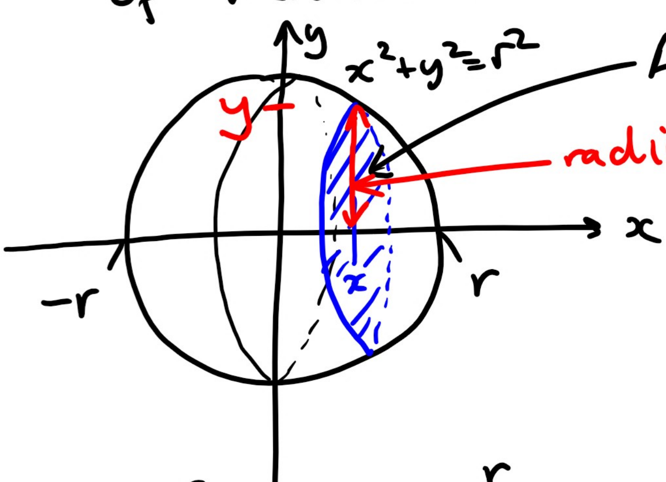
- Actual volume $= \lim_{n \to \infty} \sum_{i=1}^{n} \Delta x \, A(x_i)$

$$= \int_a^b A(x) \, dx$$

- So challenge is finding the "cross-sectional area" function $A(x)$.

<u>Example</u> Use above to find volume of sphere of radius r.

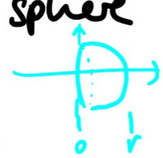
<u>Solution</u>



$A(x) = \pi y^2 = \pi (r^2 - x^2)$

radius $= \sqrt{r^2-x^2}$
$= y$

vol. of sphere $= 2 \times$ vol. of $\frac{1}{2}$ sphere



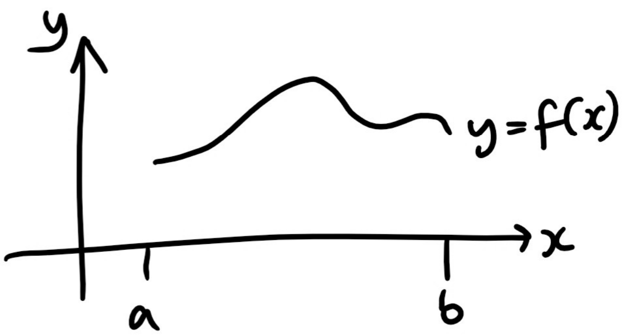
Volume : $\int_{-r}^{r} A(x) \, dx = \int_{-r}^{r} \pi (r^2 - x^2) \, dx = 2\pi \int_0^r r^2 - x^2 \, dx$

$$= 2\pi \left[r^2 x - x^3/3 \right]_0^r$$

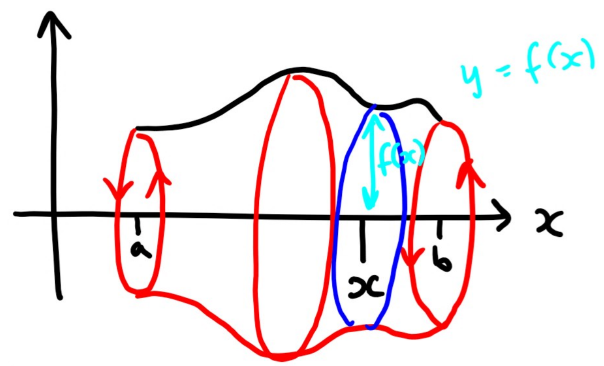
$$= 2\pi \left(r^3 - \frac{r^3}{3} - 0 \right)$$

$$= \frac{4\pi r^3}{3}$$

Solids of Rotation — solids of a special form



Rotate curve $y=f(x)$ around the x-axis to make a solid



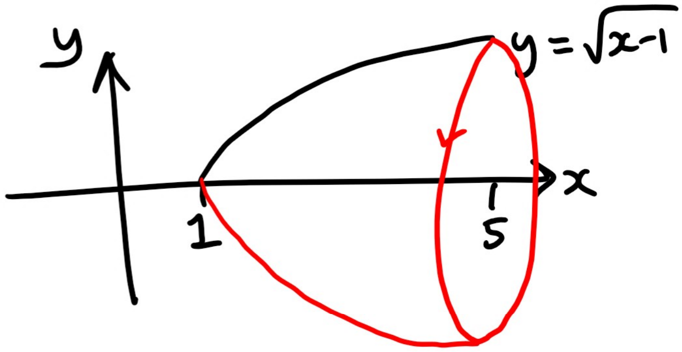
For every x, cross-section is a circle of radius $f(x)$

So $\quad A(x) = \pi (f(x))^2$

So In this case, volume $= \displaystyle\int_a^b \pi (f(x))^2 \, dx$.

Example Find volume of solid generated by rotating $y = \sqrt{x-1}$ from $x=1$ to $x=5$ around the x-axis.

Solution



$$\text{Volume} = \int_1^5 \pi\left(\sqrt{x-1}\right)^2 dx$$

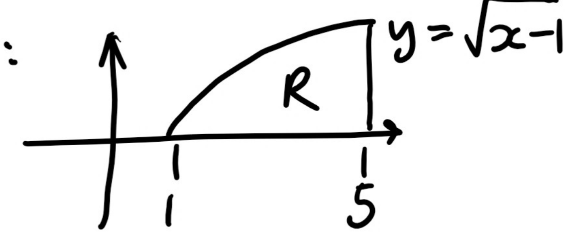
$$= \int_1^5 \pi(x-1)\,dx = \pi\left[\frac{x^2}{2} - x\right]_1^5$$

$$= \pi\left(\frac{25}{2} - 5 - \left(\frac{1}{2} - 1\right)\right)$$

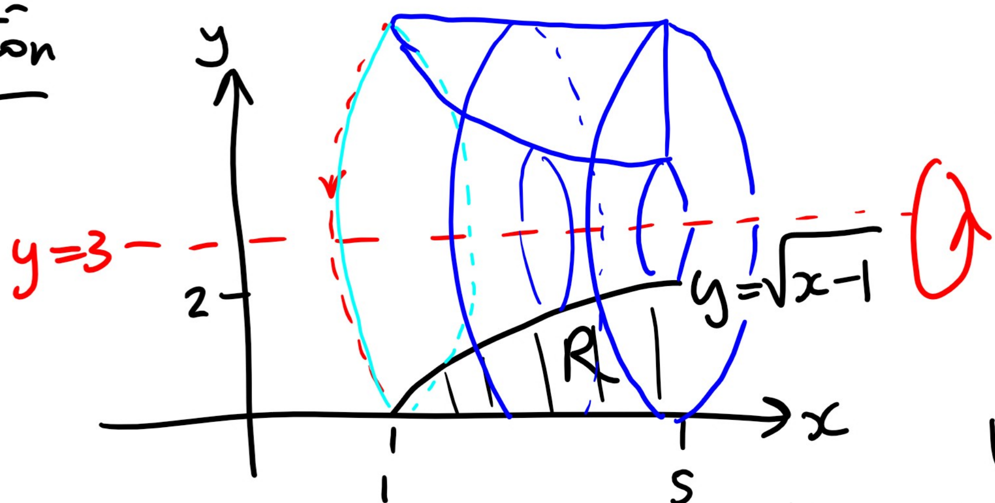
$$= \underline{\underline{8\pi}}$$

Can use this to understand more complicated solids:

__Example__ Find volume of solid generated by rotating

R :  about $y=3$:

__Solution__



← Washer-shape cross-section

We can compute the volume of outer shape & volume of "hole" using above method. T.B.C.