1ZA3 (SECTION C01) – ENGINEERING MATHEMATICS I

Lecture 30

Last time

WORK = FORCE × DISTANCE

To move an object along a straight line from $x=a$ to $x=b$ where the force needed at x is given as $f(x)$:

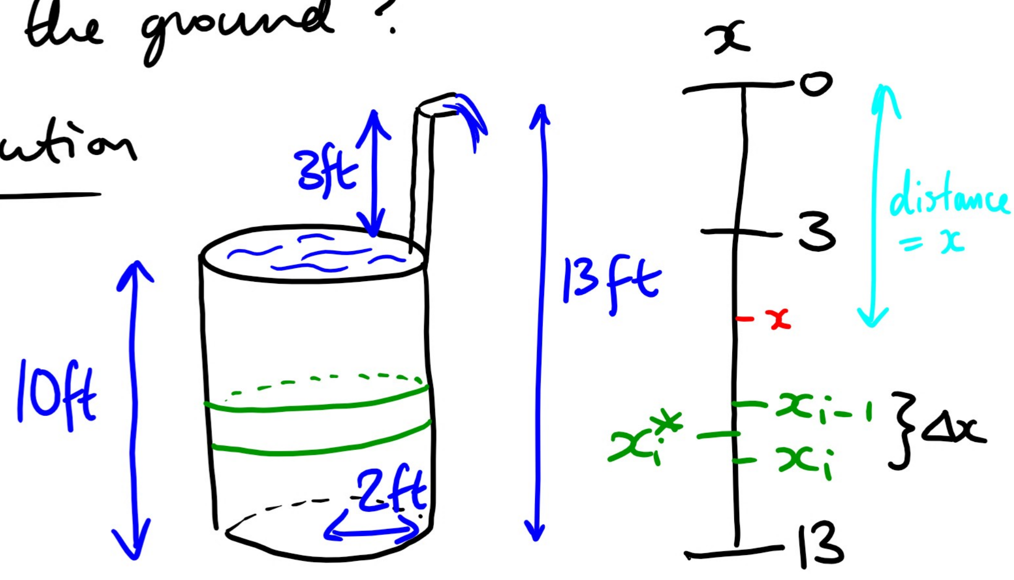
Measured in Joules = Nm or ftlb

WORK DONE: $W = \int_a^b f(x)\, dx$.

What if no nice force function $f(x)$?

Example A cylindrical tank, height 10ft, radius 2ft is filled with water. How much work is needed to pump all the water up to a spout 13ft off the ground?

Solution



- Divide $[3, 13]$ into n subintervals $[x_{i-1}, x_i]$, width $\Delta x = \frac{13-3}{n} = \frac{10}{n}$.

$\longrightarrow$ get cylinders of water of height Δx at distance $\approx x_i^*$ from spout

Volume of ith cylinder $\approx \Delta x . \pi . 2^2 = 4\pi \Delta x$.

W_i = work needed to move ith cylinder of water
↑ at $[x_{i-1}, x_i]$

= Force × distance

≈ (Volume of ith cylinder × water weight) × x_i^*

$= 4\pi \Delta x \times \underbrace{62.5\ \text{lb}/\text{ft}^3}_{\text{given this}} \times x_i^*$

$= 250\pi x_i^* \Delta x$

Need to identify from the Riemann sum what's the function is that will get integrated when we take the limit as $n \to \infty$.

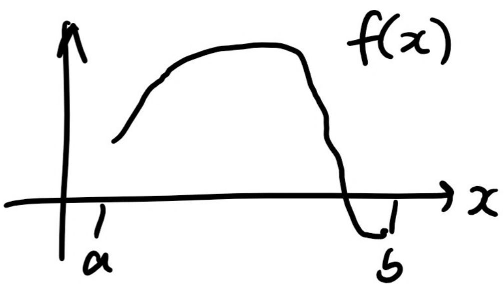
$$\text{Total work} \approx \sum_{i=1}^{n} 250\pi x_i^* \Delta x$$

$$\text{Total work} = \lim_{n\to\infty} \sum_{i=1}^{n} 250\pi x_i^* \Delta x = \int_3^{13} 250\pi x \, dx$$

$$= \left[250\pi x^2/2\right]_3^{13}$$

$$= 125(169-9)\pi = \underline{\underline{20{,}000\pi \text{ ftlb}}}$$

6.5 Average Value of a Function



Suppose we have a function for temperature, say, $f(x)$ at time x.

How do we find average temp. f_{ave} ?

Somehow have to average over all values $f(x)$ over all possible $x \in [a,b]$!

If we took temp. readings every hour, we could easily average to get an approximation.

Every minute $\longrightarrow$ better approximation

Every second $\longrightarrow$ even " " .

- Divide up $[a,b]$ into n subintervals $[x_{i-1}, x_i]$ of equal width $\Delta x = \frac{b-a}{n}$.
- For each i find $f(x_i^*)$ for a sample point in $[x_{i-1}, x_i]$.
- Average of those values $= \dfrac{f(x_1^*) + \ldots + f(x_n^*)}{n}$

↳ is an approximation to fave

Notice: $n = \dfrac{b-a}{\Delta x}$

$$= \frac{\sum_{i=1}^{n} f(x_i^*)}{(b-a)/\Delta x} = \frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*) \Delta x$$

- We define fave $= \lim_{n\to\infty} \left(\frac{1}{b-a} \sum_{i=1}^{n} f(x_i^*) \Delta x \right)$

$$= \frac{1}{b-a} \int_a^b f(x)\,dx$$

<u>Example</u> An alpine trail has elevation (in m) given by $h(x) = x^5 - 20x^3 + 350$ (x in km). What is the average elevation over the first 5km?

<u>Solution</u>

$$h_{ave} = \frac{1}{5-0}\int_0^5 x^5 - 20x^3 + 350\,dx$$

$$= \frac{1}{5}\left[\frac{x^6}{6} - 5x^4 + 350x\right]_0^5$$

$$= \ldots = \underline{245\,^5\!/_6 \text{ m}}.$$

Follow-up Q: Is there a point on the route at which we actually stand at average elevation?

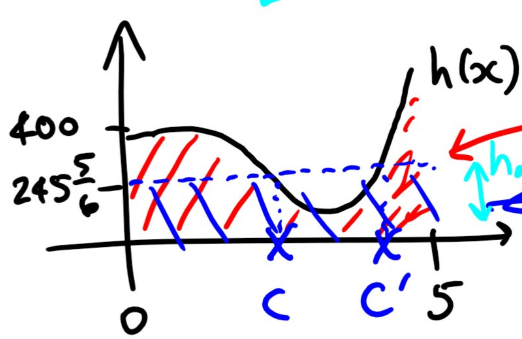
i.e. elevation $= 245\,^5\!/_6$ m

Yes: because $h(x)$ is continuous

<u>MVT for Integrals</u> If $f(x)$ is continuous on $[a,b]$, then <u>there exists</u> $c \in [a,b]$ with $f(c) = f_{ave}$.

i.e. $f(c) = \dfrac{1}{b-a}\displaystyle\int_a^b f(x)\,dx \Rightarrow \underbrace{\int_a^b f(x)\,dx}_{\text{Net area from a to b}} = \overbrace{\underbrace{f(c)}_{\text{"height"}} \cdot \underbrace{(b-a)}_{\text{width of interval } [a,b]}}^{\text{area of a rectangle}}$.

In our example



$\displaystyle\int_0^5 h(x)\,dx$ ← Area under $y = h(x)$

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$\underbrace{h(c)}_{h_{ave}} \cdot (5-0)$ ← Rectangle of height h_{ave} m region

7.1 Integration By Parts

"reverses" Product Rule.

PRODUCT RULE

$$\frac{d}{dx}\left(f(x)g(x)\right) = f(x)g'(x) + f'(x)g(x)$$

$\Rightarrow$ (In integral notation)

$$f(x)\,g(x) = \int f(x)g'(x) + f'(x)\,g(x)\,dx$$

$$= \int f(x)g'(x)\,dx + \int f'(x)g(x)\,dx$$

$$\Rightarrow \boxed{\int f(x)g'(x)\,dx = f(x)g(x) - \int f'(x)g(x)\,dx}$$

If $u = f(x)$ then $\frac{du}{dx} = f'(x) \rightarrow du = f'(x)\,dx$

$v = g(x)$ then $\frac{dv}{dx} = g'(x) \longrightarrow dv = g'(x)\,dx$

So by Substitution Rule:

$$\boxed{\int u\,dv = uv - \int v\,du}$$

Integration by Parts Formula.

Strategy for choosing u and dv:

(1) Let dv be a factor in the integrand whose antiderivative you know.

(2) Let u be a factor in the integrand that gets

"simpler" when differentiated. T.B.C. ...