1ZA3 (SECTION C01) – ENGINEERING MATHEMATICS I

Lecture 31

Last time

INTEGRATION BY PARTS

Rearranging PRODUCT RULE gives: $\int f'(x)g(x)\,dx = f(x)g(x) - \int f(x)g'(x)\,dx$

which is easier to remember as: $$\int u\,dv = uv - \int v\,du$$

u : gets simpler when you differentiate it!

dv : you can integrate it!

e.g. $x^2, x^3, \ldots$ ✓
$\ln x$, ✓ also arctan(x), arcsin(x).
e^x ✗
$\sin x$ etc. ✗
$\cosh x$ ✗
etc.

Example Find $\int x^{1/3} \ln x \, dx$.

Solution Identify u & dv.

2 choices? ① $u = x^{1/3}$ $\quad \frac{dv}{dx} = \ln x$

$\frac{du}{dx} = \frac{1}{3}x^{-2/3}$ $\quad v = ??$

② $u = \ln x \leftrightarrow \frac{dv}{dx} = x^{1/3}$

$\frac{du}{dx} = \frac{1}{x} \leftrightarrow v = \frac{3}{4}x^{4/3}$

So $$\int \underbrace{x^{1/3}}_{dv} \underbrace{\ln x}_{u}\, dx = \frac{3}{4}x^{4/3}\ln x - \int \frac{3}{4}x^{1/3}\,dx$$

$$= \frac{3}{4}x^{4/3}\ln x - \left(\frac{3}{4}\right)^2 x^{4/3} + C.$$

Example Find $\int x^2 \cos(x)\, dx$.

Solution Identify u & dv : $u = x^2$ $\quad \frac{dv}{dx} = \cos(x)$

$$\frac{du}{dx} = 2x \qquad v = \sin(x)$$

$$\int \underbrace{x^2}_{u} \underbrace{\cos x \, dx}_{dv} = x^2 \sin x - \int \underbrace{2x}_{t} \underbrace{\sin x \, dx}_{ds}$$

$$\frac{dt}{dx} = 2 \qquad s = -\cos x$$

$$= x^2 \sin x - \left(-2x\cos x - \int 2(-\cos x)\, dx \right)$$

$$= x^2 \sin x + 2x\cos x - 2\int \cos x \, dx$$

$$= x^2 \sin x + 2x \cos x - 2\sin x + C$$

$$= (x^2 - 2)\sin x + 2x\cos x + C.$$

Example Find $\int x^3 \underbrace{\cos(x^2)}_{u?\ dv?}\, dx$.

Solution

Substitute $t = x^2$

$$\frac{dt}{dx} = 2x$$

$$\int x^3 \cos(x^2)\, dx$$

$$= \int \underbrace{\cancel{x^3}^{\,2}}_{t} \cos(t) \frac{dt}{2\cancel{x}}$$

$$= \frac{1}{2} \int \underbrace{t}_{u} \underbrace{\cos(t)\, dt}_{dv} = \frac{1}{2}\left(\underbrace{t}_{u} \underbrace{\sin(t)}_{v} - \int \underbrace{\sin(t)}_{v} \underbrace{dt}_{du} \right)$$

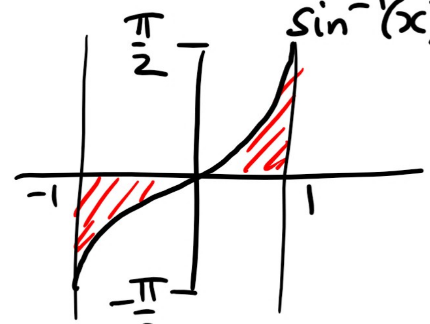
$$= \frac{1}{2}\left(t\sin(t) + \cos(t)\right) + C$$

$$= \frac{1}{2}\left(x^2\sin(x^2) + \cos(x^2)\right) + C.$$

If we have a definite integral of the form:

$\int_a^b f(x)g'(x)\,dx$ we can use F.T.C. to get

$$\hookrightarrow = \left[f(x)g(x)\right]_a^b - \int_a^b f'(x)\,g(x)\,dx$$



Odd function on a symmetric interval about 0 — must have net area $=0$.

Example Find $\int_{-1}^{1} \underbrace{\sin^{-1}(x)}_{u}\,\underbrace{dx}_{dv}$.

Solution $u = \sin^{-1}(x)$ $\qquad \frac{dv}{dx} = 1$

$$\frac{du}{dx} = \frac{1}{\sqrt{1-x^2}} \qquad v = x$$

$$\int_{-1}^{1}\sin^{-1}(x)\,dx = \left[x\sin^{-1}(x)\right]_{-1}^{1} - \underbrace{\int_{-1}^{1}\frac{x}{\sqrt{1-x^2}}\,dx}$$

Substitution : $t = 1-x^2$

$$\frac{dt}{dx} = -2x$$

$$dx = \frac{dt}{-2x}$$

$t = 1-x^2$

$$= \left[x\sin^{-1}(x)\right]_{-1}^{1} + \int_{1-(-1)^2=0}^{1-1^2=0}\frac{\cancel{x}}{\sqrt{t}}\,\frac{dt}{\cancel{-}2\cancel{x}}$$

$$= 1\left(\frac{\pi}{2}\right) - (-1)\left(-\frac{\pi}{2}\right) + \cancel{\int_0^0 \frac{1}{2\sqrt{t}}\,dt} \to 0$$

$$= \frac{\pi}{2} - \frac{\pi}{2} + 0$$

$$= 0.$$

Notice : $\sin^{-1}(x)$ on $[-1,1]$ is an odd function on a symmetric interval around 0

$\longrightarrow$ so $\displaystyle\int_{-1}^{1} \sin^{-1}(x)\,dx = 0$ $\quad\left(\text{Also } \dfrac{x}{\sqrt{1-x^2}}.\right)$ on $[-1,1]$

7.2 Trigonometric Integrals

Implication doesn't work the other way around

e.g. $\displaystyle\int_{-\pi}^{\pi} \cos x\,dx = \Big[\sin x\Big]_{-\pi}^{\pi} = \sin(\pi) - \sin(-\pi) = 0 - 0 = 0$

But $\cos x$ NOT odd!

But net area $= 0$ on $[-\pi, \pi]$

Example Find $\displaystyle\int \sin x \cos^2 x\,dx$.

Solution Substitute $u = \cos x$

$$\frac{du}{dx} = -\sin x$$

So $\displaystyle\int \sin x \cos^2 x\,dx = \int -u^2\,du = -\frac{u^3}{3} + C$

$$= -\frac{1}{3}\cos^3 x + C.$$

Example Find $\displaystyle\int \sin^3 x\,dx$.

Solution Could try subst. $u = \sin x$ $\quad \dfrac{du}{dx} = \cos x$

But not a factor. — things get messy — try it if you like & you'll see!

Better : we want a mixture of trig. functions so use trig. identities

So here use $\cos^2 x + \sin^2 x = 1$:

$$\int \sin^3 x \, dx = \int \sin x . \sin^2 x \, dx = \int \sin x . (1 - \cos^2 x) dx$$

Let $u = \cos x$

$$\frac{du}{dx} = -\sin x$$

$$dx = \frac{du}{-\sin x}$$

$$= \int \frac{\cancel{\sin x} (1 - u^2) \, du}{- \cancel{\sin x}}$$

$$= \int u^2 - 1 \, du$$

$$= \frac{u^3}{3} - u + C$$

$$= \frac{1}{3} \cos^3 x - \cos x + C .$$