1ZA3 (SECTION C01) - ENGINEERING MATHEMATICS I

Lecture 32

Last time TRIGONOMETRIC INTEGRALS

e.g. $\int \sin^3 x\,dx \overset{①}{=} \int \underbrace{\sin x}\,.\,\underbrace{\sin^2 x}\,dx \overset{②}{=} \int \sin x\,(1-\cos^2 x)\,dx \overset{③}{=} \int 1-u^2\,du$

$u = \cos x$

here $\sin(x)$

① Try to ensure there's a lone $\sin(x)$ or $\cos(x)$ factor.

② Replace any other factors that are powers of same function using $1 = \cos^2 x + \sin^2 x$.

i.e. here $\sin(x)$

③ Substitute for the other trig. function & solve.

i.e. here $\cos(x)$

Example Find $\int \cos^7 x\,dx$.

Solution

$$= \int \cos x\,.\,\cos^6 x\,dx = \int \cos x\,.\,(\cos^2 x)^3\,dx$$

$$= \int \cos x\,(1-\sin^2 x)^3\,dx$$

Now let $u = \sin x$

$du = \cos x\,dx$

$$= \int (1-u^2)^3\,du$$

→ Substitute again or just multiply out & integrate.

Example Find $\int \sin^7 x \cos^4 x\,dx$.

→ This is the general rule, but in this case, sub. doesn't help — just multiply out.

Solution

$$= \int \sin x\,.\,\sin^6 x \cos^4 x\,dx$$

Why do we not split off a $\cos x$ instead of $\sin x$? Then we'd have an odd power $\cos^3 x$ & can't use the trig. identity

$$= \int \sin x \cdot (1-\cos^2 x)^3 \cos^4 x \, dx$$

$$= -\int (1-u^2)^3 u^4 \, du \longrightarrow \text{Substitute or expand out \& solve.}$$

Subst. $u = \cos x$

$du = -\sin x \, dx$

So to evaluate $\int \sin^n x \cos^m x \, dx$ and

- if n odd, split off $\sin x$: $\sin x . \underbrace{\sin^{n-1} x} \cos^m x$

  & replace $\sin^{n-1} x$ with $(1-\cos^2 x)^{\frac{n-1}{2}}$ (n-1 now an even # so this makes sense!)

  & substitute $u = \cos x$.

- if m odd, split off $\cos x$: $\cos x \underbrace{\cos^{m-1} x} \sin^n x$

  & replace $\cos^{m-1} x$ with $(1-\sin^2 x)^{\frac{m-1}{2}}$ m-1 now even!

  & substitute $u = \sin x$

- if m & n are both odd you choose!

- If m & n are both even, use $\frac{1}{2}$-angle formulae: $\sin^2 x = \frac{1}{2}(1-\cos 2x)$

$$\cos^2 x = \frac{1}{2}(1+\cos 2x)$$

<u>Example</u> Find $\int \cos^2 x \sin^4 x \, dx$.

Solution

$$= \int \frac{1}{2}(1+\cos 2x)\left(\left(\tfrac{1}{2}\right)(1-\cos 2x)\right)^2 dx$$

$$= \frac{1}{8}\int \underbrace{(1+\cos 2x)(1-\cos 2x)^2}_{\text{Expand as a cubic in } \cos 2x} dx$$

$$= \frac{1}{8}\int 1 - \cos 2x - \underbrace{\cos^2 2x} + \underbrace{\cos^3 2x}_{\text{substitute } u = 2x} \, dx$$

This puts us in odd case above

$$= \frac{1}{8}\int 1 - \cos 2x - \frac{1}{2}(1+\cos 4x)\, dx$$

$u = 2x$ so $\frac{du}{dx} = 2$

$$+ \frac{1}{16}\int \cos^3 u \, du$$

$$= \frac{1}{16}\int \cos u \cos^2 u \, du = \frac{1}{16}\int 1 - v^2 \, dv$$

Method above ($v = \sin u$, $\frac{dv}{du} = \cos u$)

$$= \ldots \frac{1}{16}\left(x - \sin 2x - \frac{1}{2}\sin 4x + v - \frac{1}{3}v^3\right)$$

$$= \frac{1}{16}\left(x - \cancel{\sin 2x} - \frac{1}{2}\sin 4x + \cancel{\sin 2x} - \frac{1}{3}\sin^3 2x\right)$$

This idea also works with $\int \tan^n x \sec^m dx$

Now we want to split off $\sec^2 x$ or $\sec x \tan x$

because these are the derivatives: (think Substitution Rule) $= \frac{d}{dx}\tan x$ $= \frac{d}{dx}\sec x$.

& use $\sec^2 x = 1 + \tan^2 x$.

Example Find $\int \tan^5 x \sec^3 x \, dx$.

Solution Could try splitting off $\sec^2 x$

$$\to \int \tan^5 x \,.\, \underbrace{\sec x}_{??} \,.\, \sec^2 x \, dx$$

?? Trig identity not useful.

So instead $\displaystyle\int \tan^5 x \sec^3 x\,dx = \int \underbrace{\tan^4 x \sec^2 x}\,\underbrace{(\tan x \sec x)\,dx}_{=du \text{ if } u=\sec x}$

so get rid of $\tan x$ using trig. identity (so we only have $\sec x$ left & can sub. in $u=\sec x$.

$$= \int (\sec^2 x - 1)^2 \cdot \sec^2 x \cdot \tan x \sec x\,dx$$

Now sub. $u=\sec x$

$$= \int (u^2-1)^2 u^2\,du \longrightarrow \text{Expand out \& integrate.}$$

<u>Example</u> Find $\displaystyle\int \tan^2 x \sec^4 x\,dx$

<u>Solution</u> Split off $\sec^2 x$ $\hookrightarrow$ $\displaystyle = \int \tan^2 x \underbrace{\sec^2 x}_{1+\tan^2 x}\,\underbrace{\sec^2 x\,dx}_{du}$

du if $u=\tan x$

(Want to be left only with $\tan x$ since we want this substitution)

$u=\tan x$

$$= \int u^2(1+u^2)\,du$$

$$= \int u^2 + u^4\,du = \frac{1}{3}u^3 + \frac{1}{5}u^5 + C$$

$$= \frac{1}{3}\tan^3 x + \frac{1}{5}\tan^5 x + C.$$

So to evaluate $\displaystyle\int \tan^n x \sec^m x\,dx$

- if m even, split off $\sec^2 x$, rewrite $\sec^{m-2} x$ as $(1+\tan^2 x)^{\frac{m-2}{2}}$ & substitute $u = \tan x$

  m even $\Rightarrow$ $m-2$ even

- if n odd, split off $\sec x \tan x$, rewrite $\tan^{n-1} x$ as $(\sec^2 x - 1)^{\frac{n-1}{2}}$ & substitute $u = \sec x$.

  n odd $\Rightarrow$ $n-1$ even

Other cases require creativity! Perhaps need:

$$\int \tan x \, dx \overset{\text{subs. } u=\cos x}{=} \ln|\sec x| + C$$

$$\int \sec x \, dx = \ln|\sec x + \tan x| + C$$

See textbook or check by differentiating RHS

<u>Example</u> Find $\int \sin 6x \cos 5x \, dx$.

<u>Solution</u> Need formula: (1) $\sin A \cos B = \frac{1}{2}(\sin(A-B) + \sin(A+B))$

$$= \int \frac{1}{2}(\sin(x) + \sin(11x)) \, dx$$

$$= -\frac{1}{2}\left(\cos x + \frac{1}{11}\cos(11x)\right) + C.$$

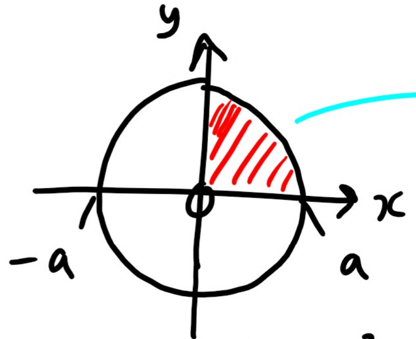
Remember also: (2) $\sin A \sin B = \frac{1}{2}(\cos(A-B) - \cos(A+B))$

(3) $\cos A \cos B = \frac{1}{2}(\cos(A-B) + \cos(A+B))$.

7.3 Trigonometric Substitution

Example Use integration to find the area of a circle of radius $a > 0$.

Solution



$$\int_0^a \underbrace{\sqrt{a^2 - x^2}}_{??} \, dx = \text{Area of } \tfrac{1}{4}\text{-circle}$$

$y^2 + x^2 = a^2$

$y = \sqrt{a^2 - x^2}$

(on top half of circle)

?? No obvious candidate for substitution

Trick Substitute $\theta = \sin^{-1}\left(\frac{x}{a}\right) \longrightarrow a\sin\theta = x$

Why?? (It works)

$$\frac{d\theta}{dx} = \frac{1}{a\sqrt{1 - \left(\frac{x}{a}\right)^2}} = \frac{1}{a\sqrt{1 - \sin^2\theta}} = \frac{1}{|a\cos\theta|}$$

That was hard work to get dx in terms of θ! Easier to notice $x = a\sin\theta$ so $dx = a\cos\theta \, d\theta$

$$dx = |a\cos\theta| \, d\theta$$

Don't really need these absolute value signs as $\sin^{-1}$ only gives values θ in $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ where $\cos\theta \geq 0$.

$\frac{\pi}{2} = \sin^{-1}\left(\frac{a}{a}\right)$

$0 = \sin^{-1}\left(\frac{0}{a}\right)$

$$\int_0^{\pi/2} \sqrt{a^2 - a^2\sin^2\theta} \, |a\cos\theta| \, d\theta = \int_0^{\pi/2} a\cos\theta \, |a\cos\theta| \, d\theta$$

$$= \int_0^{\pi/2} a^2\cos^2\theta \, d\theta$$

$$= \frac{1}{2}\int_0^{\pi/2} a^2(1 + \cos 2\theta) \, d\theta = \frac{\pi a^2}{4}$$

$$= \frac{1}{2}\left[a^2\theta + \frac{a^2}{2}\sin 2\theta\right]_0^{\pi/2} \ldots$$

Area of whole circle

$$\Rightarrow \text{Area} = 4 \times \overbrace{\frac{\pi a^2}{4}}^{\text{Area of } \frac{1}{4}\text{-circle}} = \pi a^2.$$