1ZA3 (SECTION C01) - ENGINEERING MATHEMATICS I

Lecture 33

Last time TRIGONOMETRIC SUBSTITUTION

RULE

① To simplify $\sqrt{a^2 - x^2}$ in an integral expression,*

substitute $\theta = \sin^{-1}(\frac{x}{a})$ or better said, do an

ONLY VALID for $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$.

inverse substitution : $x = a\sin\theta$ ($dx = a\cos\theta$)

& use $1 - \sin^2\theta = \cos^2\theta$: $\sqrt{a^2 - x^2} = a\cos\theta$

* if other methods don't help

② To simplify $\sqrt{a^2 + x^2}$, substitute $\theta = \tan^{-1}(\frac{x}{a})$

i.e. $x = a\tan\theta$

valid for $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$

$\sqrt{a^2 + a^2\tan^2\theta} = \sqrt{a^2\underbrace{(1+\tan^2\theta)}_{\sec^2\theta}}$

$= a\sec\theta$

using $1+\tan^2\theta = \sec^2\theta$

$\frac{dx}{d\theta} = a\sec^2\theta$

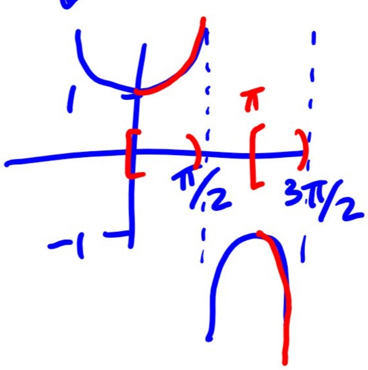
③ To simplify $\sqrt{x^2 - a^2}$, substitute $\theta = \sec^{-1}(\frac{x}{a})$

???

i.e. $x = a\sec\theta$

$\frac{dx}{d\theta} = a\sec\theta\tan\theta$

$$\sqrt{a^2\sec^2\theta - a^2} = \sqrt{a^2\underbrace{(\sec^2\theta - 1)}_{\text{using } \sec^2\theta - 1 = \tan^2\theta}} = a\tan\theta$$



valid for θ in $[0, \frac{\pi}{2})$ or $[\pi, \frac{3\pi}{2})$

Example Find $\displaystyle\int \frac{dx}{\sqrt{x^2+4x-5}}$.

Solution

This does not immediately have the required form but complete the square:

$$x^2 + 4x - 5 = (x + \overset{1/2}{2})^2 - 9$$

$$\text{our integral} = \int \frac{dx}{\sqrt{(x+2)^2 - 9}} = \int \frac{du}{\sqrt{u^2 - 3^2}}$$

Sub. $u = x + 2$ i.e. define u in terms of x
$du = dx$

Case 3 sub. $u = 3\sec\theta$ i.e. define θ implicitly in terms of u
$du = 3\sec\theta\tan\theta\, d\theta$

$$= \int \frac{3\sec\theta\tan\theta\, d\theta}{\sqrt{3\sec^2\theta - 3^2}}$$

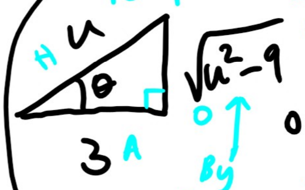
$$= \int \frac{\cancel{3}\sec\theta\cancel{\tan\theta}}{\cancel{3}\cancel{\tan\theta}}\, d\theta$$

$$= \int \sec\theta \, d\theta = \ln|\sec\theta + \tan\theta| + C$$

Unwind the trig. sub.

$$= \ln\left|\frac{u}{3} + \frac{\sqrt{u^2-9}}{3}\right| + C$$

To find $\tan\theta$ (when we know $\sec\theta = u/3$)



↑ By Pythagoras or trig. id. $\tan^2\theta = \sec^2\theta - 1$

Unwind the sub.

$$= \ln\left|\frac{x+2}{3} + \frac{\sqrt{(x+2)^2-9}}{3}\right| + C$$

$$= \ln\left|\frac{x+2+\sqrt{x^2+4x-5}}{3}\right| + C$$

$$= \ln\left|x+2+\sqrt{x^2+4x-5}\right| \cancel{- \ln 3 + C} \; C$$

$$= \ln\left|x+2+\sqrt{x^2+4x-5}\right| + C \leftarrow \text{arb. constants}$$

Try to absorb all constant terms into C.

7.4 Integration of Rational Functions by Partial Fractions

Rational Function $f(x) = \dfrac{P(x)}{Q(x)}$ ← polynomials

We want $\displaystyle\int f(x)\,dx = \int \frac{P(x)}{Q(x)}\,dx$

Step 1 If degree(P) $\geq$ degree(Q) ($f(x)$ "improper")

do long division :

get $\dfrac{P(x)}{Q(x)} = S(x) + \underbrace{\dfrac{R(x)}{Q(x)}}_{\substack{\text{degree}(R) \\ < \text{degree}(Q) \\ (\text{"proper"})}}$ ← remainder

polynomial

We know how to integrate this!

Example $\dfrac{P(x)}{Q(x)} = \dfrac{2x^4+4x^3-11x^2-9x+16}{x^2+2x-3}$

$$
\begin{array}{r l}
 & 2x^2-5 \quad \leftarrow S(x) \\
\underbrace{x^2+2x-3}_{Q(x)} \big) & 2x^4+4x^3-11x^2-9x+16 \\
 & 2x^4+4x^3-6x^2 \\ \hline
 & -5x^2-9x+16 \\
 & -5x^2-10x+15 \\ \hline
 & x+1 \quad \leftarrow R(x)
\end{array}
$$

$$\frac{P(x)}{Q(x)} = \underbrace{2x^2-5}_{S(x)} + \frac{x+1}{x^2+2x-3} \quad \begin{array}{l} \leftarrow R(x) \\ \leftarrow Q(x). \end{array}$$

Step 2 Factor $Q(x)$ into irreducible factors.

The "Fundamental Theorem of Algebra" tells us all these factors look like $ax+b$ (linear) or $ax^2 + bx + c$ (irreducible quadratic)

$\hookrightarrow b^2 - 4ac < 0$

Example $Q(x) = x^2 + 2x - 3 = (x+3)(x-1)$

Step 3 4 cases, depending on factors of $Q(x)$:

Case I $Q(x)$ has distinct linear factors only.

Case II $Q(x)$ has linear factors only, but at least one repeats. e.g. $(x+1)^2$

Case III $Q(x)$ has at least one irreducible quadratic factor, but no repeated irr. quad. factors

Case IV $Q(x)$ has at least one repeated irreducible quadratic factor.

Case I $Q(x)$ has distinct linear factors only

$$= (a_1x + b_1)(a_2x + b_2) \cdots (a_kx + b_k)$$

We found these factors of $Q(x)$ in Step 2.

Then $$\frac{R(x)}{Q(x)} = \underbrace{\frac{A_1}{a_1x+b_1} + \frac{A_2}{a_2x+b_2} + \dots + \frac{A_k}{a_kx+b_k}}$$

for some #s $A_1, \dots, A_k$ (which we will need to find)

$\longrightarrow$ This we integrate

Partial Fraction Expression.

How to find this? Solve for $A_1, \ldots, A_k$.

(We know the $(a_i + b_i x)$ terms once we factor $Q(x)$, so all that's left to find are these #s)

<u>Example</u> From earlier example $\dfrac{R(x)}{Q(x)} = \dfrac{x+1}{(x+3)(x-1)}$

We want to write it as $= \dfrac{A}{x+3} + \dfrac{B}{x-1}$

So we need to find A, B.

Multiply through by $Q(x)$:

(gives a much nicer equation to work with)

$$x+1 = A(x-1) + B(x+3).$$

We could compare coefficients: $x+1 = Ax - A + Bx + 3B = (A+B)x + (3B-A)$

So $A+B=1$, $3B-A=1$

$\Rightarrow 4B = 2 \Rightarrow B = \frac{1}{2} \Rightarrow A = \frac{1}{2}$.

Trick: choose clever x-values:

(After all, the above equation is true for every x.)

↳ This helps with linear factors & getting rid of terms to make simple equations for the unknowns

$x = 1$: $1+1 = B(1+3) \Rightarrow B = \frac{1}{2}$

↳ makes the $(x-1)$ term go away on RHS

$x = -3$: $-3+1 = A(-3-1) \Rightarrow A = \frac{1}{2}$

↳ makes the $x+3$ term go away on RHS

Thus $\dfrac{R(x)}{Q(x)} = \underbrace{\dfrac{1}{2(x+3)} + \dfrac{1}{2(x-1)}}$.

$\dfrac{R(x)}{Q(x)} = \dfrac{x+1}{(x+3)(x-1)}$

↳ Check it really gives the LHS $\frac{R(x)}{Q(x)}$ back!