1ZA3 (SECTION C01) - ENGINEERING MATHEMATICS I

Lecture 34

Last time $\int \frac{P(x)}{Q(x)}\,dx$ by Partial Fractions

Step 1 Write $\frac{P(x)}{Q(x)} = S(x) + \boxed{\frac{R(x)}{Q(x)}}$ using e.g. long division

polynomial   remainder; degree(R) < degree(Q)

Step 2 Factorize $Q(x)$ into $ax+b$ and ax^2+bx+c factors

$b^2-4ac<0$

Step 3 Case I If $Q(x)$ has only distinct, linear factors, write

PARTIAL FRACTION EXPRESSION $\rightarrow$ $\boxed{\frac{R(x)}{Q(x)}} = \frac{A_1}{a_1x+b_1} + \dots + \frac{A_k}{a_kx+b_k}$ ← solve for $A_1, \dots, A_k$ ← factors of $Q(x)$

Our Example continued:

$$\frac{P(x)}{Q(x)} = \frac{2x^4+4x^3-11x^2-9x+16}{x^2+2x-3} = S(x) + \frac{R(x)}{Q(x)}$$

$$= \overbrace{2x^2-5} + \frac{x+1}{\underbrace{x^2+2x-3}}$$

$$= \frac{x+1}{(x-1)(x+3)} = \frac{1}{2(x-1)} + \frac{1}{2(x+3)}$$

$$\int \frac{P(x)}{Q(x)}\,dx = \int 2x^2-5 + \frac{1}{2\underbrace{(x-1)}_{u=x-1}} + \frac{1}{2(x+3)}\,dx$$

$$= \frac{2}{3}x^3 - 5x + \frac{1}{2}\ln|x-1| + \frac{1}{2}\ln|x+3| + C$$

Case II $Q(x)$ has linear factors only, but at least one repeats.

If $(a_i x + b_i)$ is repeated r times, replace

$\frac{A_i}{a_i x + b_i}$ in Partial Fraction Expression from Case I

with
$$\frac{A_{i,1}}{(a_i x + b_i)} + \frac{A_{i,2}}{(a_i x + b_i)^2} + \dots + \frac{A_{i,r}}{(a_i x + b_i)^r}$$

Then solve for all the As & integrate.

Example Find $\displaystyle\int \frac{3x-2}{x^2(x-1)^2(x+2)}\,dx$.

Solution Notice already proper with $Q(x)$ factorized so no need for Steps 1 & 2.

Case II, Step 3: Write

$$\frac{3x-2}{x^2(x-1)^2(x+2)} = \frac{A_1}{x} + \frac{A_2}{x^2} + \frac{B_1}{x-1} + \frac{B_2}{(x-1)^2} + \frac{C}{x+2}$$

Multiply by $Q(x)$:

$$3x-2 = A_1 x(x-1)^2(x+2) + A_2(x-1)^2(x+2) + B_1 x^2(x-1)(x+2)$$
$$+ B_2 x^2(x+2) + C x^2(x-1)^2$$

2 choices: ① multiply out & compare coefficients of x^4, x^3, x^2, x, constant

② "Cover-up": plug in clever x-values:

$$\left.\begin{array}{lll} x=0 & : & A_2 = -1 \\ x=1 & : & B_2 = \frac{1}{3} \\ x=-2 & : & C = -2/9 \end{array}\right\}$$

Now we need to plug in 2 more x-values to get A_1, B_1 – we choose "easy" ones (& we use)

$x=-1$: $\quad -5 = -4A_1 - 4 - 2B_1 + \frac{1}{3} - \frac{8}{9}$ (Ex.!)

$$\longrightarrow 2A_1 + B_1 = \frac{2}{9}$$

$x=2$: $\quad 4 = 8A_1 - 4 + 16B_1 + \frac{16}{3} - \frac{8}{9}$ (Ex!)

$$\longrightarrow A_1 + 2B_1 = \frac{4}{9}$$

$$\longrightarrow A_1 = 0 \quad \text{and} \quad B_1 = \frac{2}{9}$$

All together: $$\int \frac{3x+2}{x^2(x-1)^2(x+2)}\,dx = \int -\frac{1}{x^2} + \frac{2}{9(x-1)} + \frac{1}{3(x-1)^2} - \frac{2}{9(x+2)}\,dx$$

$$= \frac{1}{x} + \frac{2}{9}\ln|x-1| - \frac{1}{3(x-1)} - \frac{2}{9}\ln|x+2| + C$$

Case III $Q(x)$ contains at least one irred. quadratic factor, but no irred. quadratic factor repeats.

For each factor:

if linear, treat it as in Cases I & II;

if quadratic ax^2+bx+c, include

$$\frac{Ax+B}{ax^2+bx+c}$$ in partial fraction expression for $\frac{R(x)}{Q(x)}$.

Then integrate.

Example Find $\displaystyle\int \frac{x^2-3x+3}{(x+1)^2(2x^2-3x+2)}\,dx$

$b^2-4ac = 9-4.2.2 = -7<0$

So proper & $Q(x)$ fully factorized so jump to Step 3. We're in Case III.

Set

$$\frac{x^2-3x+3}{(x+1)^2(2x^2-3x+2)} = \frac{A}{x+1} + \frac{B}{(x+1)^2} + \frac{Cx+D}{2x^2-3x+2}$$

Multiply by $Q(x)$:

$$x^2 - 3x + 3 = A(x+1)(2x^2-3x+2) + B(2x^2-3x+2) + (Cx+D)(x+1)^2$$

Multiply out & compare coefficients:

$$x^2 - 3x + 3 = (2A+C)x^3 + (-A+2B+2C+D)x^2 + (-A-3B+C+2D)x + (2A+2B+D)$$

Solve:

$$\left.\begin{aligned} 2A + C &= 0 \\ -A + 2B + 2C + D &= 1 \\ -A - 3B + C + 2D &= -3 \\ 2A + 2B + D &= 3 \end{aligned}\right\} \quad A = \frac{2}{7},\ B = 1,\ C = -\frac{4}{7},\ D = \frac{3}{7}.$$

So

$$\int \frac{x^2 - 3x + 3}{(x+1)^2(2x^2-3x+2)}\,dx =$$

$$\int \frac{2}{7(x+1)}\,dx + \int \frac{1}{(x+1)^2}\,dx + \int \frac{3-4x}{7\underbrace{(2x^2-3x+2)}}\,dx$$

$u = 2x^2 - 3x + 2$
$du = 4x - 3$

$$= \frac{2}{7}\ln|x+1| - \frac{1}{x+1}$$

$$+ \int -\frac{1}{7u}\,du \longrightarrow -\frac{1}{7}\ln|u| + c$$

$$= \frac{2}{7}\ln|x+1| - \frac{1}{x+1} - \frac{1}{7}\ln|2x^2 - 3x + 2| + C.$$