1ZA3 (SECTION C01) – ENGINEERING MATHEMATICS I

Lecture 35

Last time $\displaystyle\int \frac{P(x)}{Q(x)}\,dx$ using PARTIAL FRACTIONS (ctd)

From $\dfrac{P(x)}{Q(x)} = S(x) + \dfrac{R(x)}{Q(x)}$ } we form a Partial Fraction Expression for $\dfrac{R(x)}{Q(x)}$ depending on factors of $Q(x)$.

e.g. Case III

$$(ax+b)^r \to \frac{A_1}{ax+b} + \dots + \frac{A_r}{(ax+b)^r} \text{ and } ax^2+bx+c \longrightarrow \frac{Ax+B}{ax^2+bx+c}.$$

Case IV $Q(x)$ has at least one repeated irred. quad. factor.

If $(ax^2+bx+c)^r$ is a factor, include

$$\frac{A_1x+B_1}{ax^2+bx+c} + \frac{A_2x+B_2}{(ax^2+bx+c)^2} + \dots + \frac{A_rx+B_r}{(ax^2+bx+c)^r}$$

in the Partial Fraction Expression for $Q(x)$, & integrate.

Example Find $\displaystyle\int \frac{3x^4-16x^3+42x^2-54x+23}{(x-3)(x^2-2x+2)^2}\,dx$

Solution

$$= \frac{A}{x-3} + \frac{Bx+C}{x^2-2x+2} + \frac{Dx+E}{(x^2-2x+2)^2}$$

Use computer package! $= \dfrac{2}{x-3} + \dfrac{x-3}{x^2-2x+2} + \dfrac{3x+1}{(x^2-2x+2)^2}$

Now $\displaystyle\int \frac{2}{x-3}\,dx = \boxed{2\ln|x-3| + C}$ ✓

Next $\displaystyle\int \underbrace{\frac{x-3}{x^2-2x+2}}_{\text{Split into}}\,dx$

This is what we would have wished for, so make the wish come true!

Split into $\overset{K}{\text{const.}}\,\dfrac{(x^2-2x+2)'}{x^2-2x+2} + \dfrac{\overset{L}{\text{const.}}}{x^2-2x+2}$

(We want to sub. $u = x^2-2x+2$, $du = (x^2-2x+2)'\,dx$)

So find K, L with $x-3 = K(x^2-2x+2)' + L$

$$x-3 = K(2x-2) + L$$

Compare coeffs : $K = 1/2$, $L = -2$

Then
$$\int \frac{x-3}{x^2-2x+2}\,dx = \frac{1}{2}\int \underbrace{\frac{2x-2}{x^2-2x+2}}_{u=x^2-2x+2}\,dx - 2\int \frac{dx}{x^2-2x+2}$$

$$= \frac{1}{2}\int \frac{du}{u}$$

$$= \frac{1}{2}\ln|u| + C = \boxed{\frac{1}{2}\ln|x^2-2x+2| + C}$$

$$= -2\int \frac{dx}{(x-1)^2+1}$$

(complete the square)

$$\underset{\substack{\text{sub.}\\ v=x-1\\ dv=dx}}{=} -2\int \frac{dv}{v^2+1}$$

Finally $\displaystyle\int \frac{3x+1}{(\underbrace{x^2-2x+2}))^2}\,dx$

$= -2\arctan(v) + C$

By knowing this!!

$= \boxed{-2\arctan(x-1) + C}$

Unwinding the sub.

Want to sub. $u = x^2-2x+2$ again.

Want to split into 2 pieces $\displaystyle = \frac{\text{const.}\,\overbrace{(2x-2)}^{(x^2-2x+2)'}}{(x^2-2x+2)^2} + \frac{\text{const.}}{(x^2-2x+2)^2}$

just as above

M

N

i.e. find M, N with

Again, this is what we would have wished for, so make it happen.

$$3x+1 = M(2x-2) + N \longrightarrow M = \frac{3}{2},\ N = 4.$$

So we ~~get~~

$$\int \frac{3x+1}{(x^2-2x+2)^2}\,dx = \frac{3}{2}\int \frac{2x-2}{(\underbrace{x^2-2x+2})^2}\,dx + 4\int \frac{dx}{(\underbrace{x^2-2x+2})^2}$$

$u = x^2-2x+2$

complete the square

$$= \frac{3}{2}\int \frac{du}{u^2}$$

$$= -\frac{3}{2u} + C$$

$$= \boxed{-\frac{3}{2(x^2-2x+2)} + C}$$

$$= 4\int \frac{dx}{((\underbrace{x-1})^2+1)^2}$$

$v = x-1$

$dv = dx$

$$= 4\int \frac{dv}{(v^2+1)^2}$$

When you see this, think trig. sub., even if no $\sqrt{}$ around (as long as there's no easier way!)

$$\underset{v=\tan\theta}{=} 4\int \frac{\sec^2\theta}{(\tan^2\theta+1)^2}\,d\theta$$

$$= 4\int \frac{\sec^2\theta}{(\sec^2\theta)^2}\,d\theta$$

$$= 4\int \frac{1}{\sec^2\theta}\,d\theta$$

$$= 4\int \cos^2\theta\,d\theta$$

$$= \frac{4}{2}\int \cos 2\theta + 1\,d\theta$$

Exercise

Q. What is $\sin 2\theta$ when $v = \tan\theta$?

$$= 2\theta + \sin 2\theta + C$$

$$= 2\arctan(v) + \frac{2v}{v^2+1} + C$$

$$= \boxed{2\arctan(x-1) + \frac{2(x-1)}{x^2-2x+2} + C}$$

Phew!

ANSWER

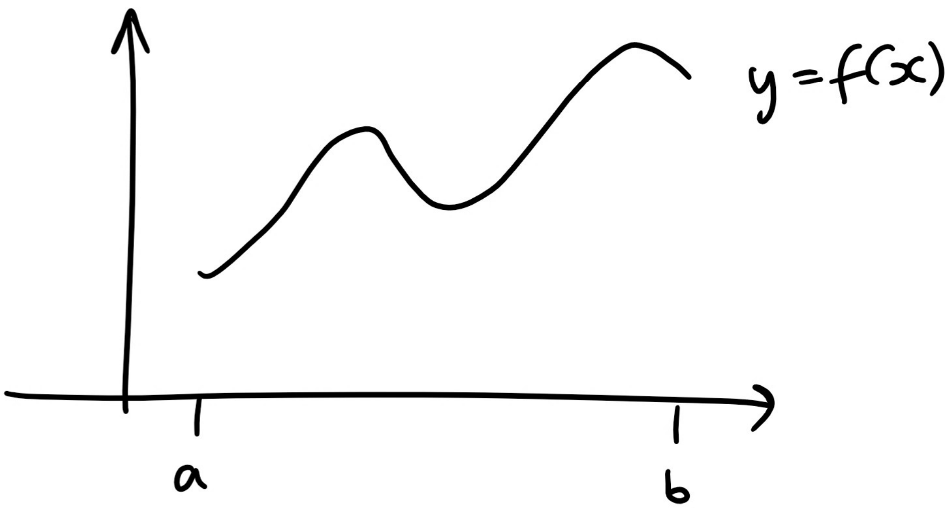
$$2\ln|x-3| + \frac{1}{2}\ln|x^2-2x+2| \cancel{- 2\arctan(x-1)}$$

$$+ \frac{4x-7}{2(x^2-2x+2)} \cancel{+ 2\arctan(x-1)} + C$$

For the numerator of x^2-2x+2 : $-3/2 + 2(x-1) = -\frac{3}{2} + \frac{4x-4}{2}$

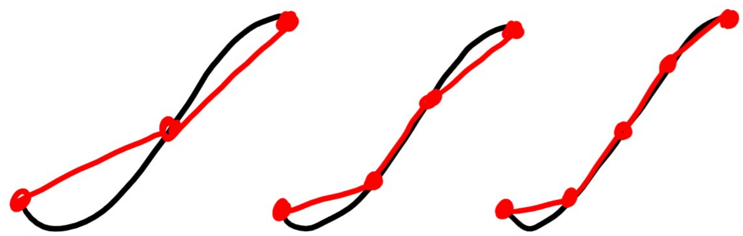
One last application of integration!

8.1 Arc length



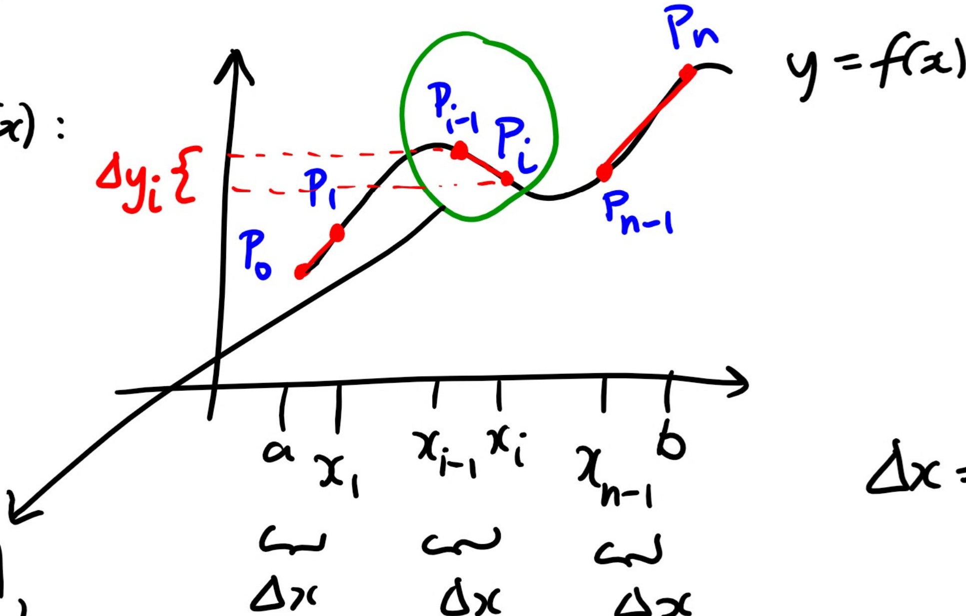
How do we find the "arc length" of $y = f(x)$?

Approx. with line segments:



The more line segments, the better the approximation.

So for $y = f(x)$:



$$\Delta x = \frac{b-a}{n}$$

What is $|P_{i-1}P_i|$, the length of the line segment from P_{i-1} to P_i?

$$|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + (\Delta y_i)^2}$$

depends on i

The amount y changes between P_{i-1} and P_i

What is Δy_i?

$$\overset{\shortparallel}{f(x_i) - f(x_{i-1})}.$$

By M.V.T. there is <u>some</u> $c_i \in [x_{i-1}, x_i]$ with

$$f'(c_i) = \frac{f(x_i) - f(x_{i-1})}{x_i - x_{i-1}}$$

So $\Delta y_i = f(x_i) - f(x_{i-1}) = f'(c_i)(x_i - x_{i-1}) = f'(c_i)\Delta x.$

So $$|P_{i-1}P_i| = \sqrt{(\Delta x)^2 + f'(c_i)^2(\Delta x)^2}$$

$$= \left(\sqrt{1 + f'(c_i)^2}\right)\Delta x.$$

T.B.C.