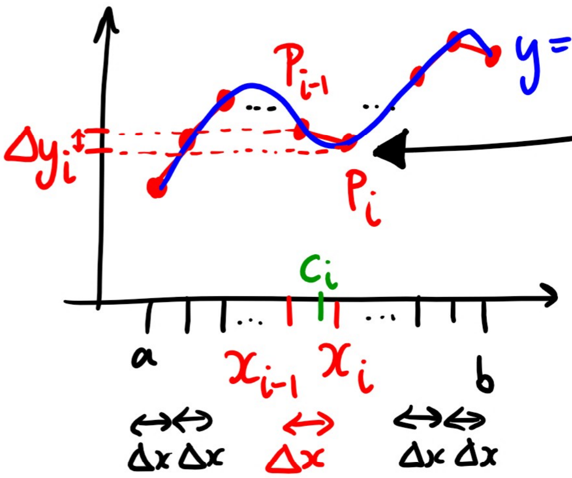
1ZA3 (SECTION C01) - ENGINEERING MATHEMATICS I

Lecture 36

Last time

So far: **Arc Length** – What is the length of $y=f(x)$?



$|P_{i-1}P_i|$ = length of line segment $P_{i-1} \to P_i$

$$= \sqrt{1+(f'(c_i))^2}\cdot\Delta x$$

$$\left[\text{where } \underset{\text{(M.V.T.)}}{f'(c_i)} = \frac{f(x_i)-f(x_{i-1})}{x_i - x_{i-1}} = \frac{\Delta y_i}{\Delta x}\right]$$

So arc length of $y=f(x)$ from $x=a$ to $x=b$

$$\approx \sum_{i=1}^{n} |P_{i-1}P_i|$$

$$= \sum_{i=1}^{n} \sqrt{1+(f'(c_i))^2}\,\Delta x$$

So length of $y=f(x)$ from $x=a$ to $x=b$

$$= \lim_{n\to\infty} \sum_{i=1}^{n} \sqrt{1+(f'(c_i))^2}\,\Delta x$$

↑ acts as a sample point

$$= \int_a^b \sqrt{1+(f'(x))^2}\,dx$$

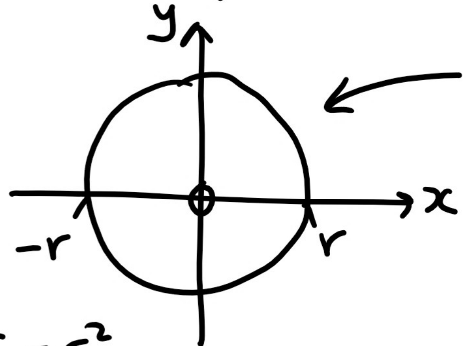
if $f'(x)$ is continuous.

Notice that x_{i-1}, c_i and x_i all end up in the same place as $n\to\infty$. $\underset{x_{i-1}}{|}\,\underset{c_i}{|}\,\underset{x_i}{|}$ (all squished together)

$$\text{OR} = \int_a^b \sqrt{1+\left(\frac{dy}{dx}\right)^2}\, dx \quad \text{(if you prefer)}$$

<u>Example</u> Show why the circumference of a circle of radius r is $2\pi r$.

<u>Solution</u>



$x^2+y^2=r^2$

top $\frac{1}{2}$: $y=\sqrt{r^2-x^2} = (r^2-x^2)^{1/2}$.

$$\frac{dy}{dx} = \frac{1}{\cancel{2}}(-\cancel{2}x)(r^2-x^2)^{-1/2}$$

$$= \frac{-x}{\sqrt{r^2-x^2}}$$

Circum. of top $\frac{1}{2}$ of circle

$$= \int_{-r}^{r} \sqrt{1+\left(\frac{-x}{\sqrt{r^2-x^2}}\right)^2}\, dx = \int_{-r}^{r}\sqrt{1+\frac{x^2}{r^2-x^2}}\, dx$$

$$= \int_{-r}^{r}\sqrt{\frac{r^2\cancel{-x^2}\ \cancel{+x^2}}{r^2-x^2}}\, dx = r\int_{-r}^{r}\frac{dx}{\sqrt{r^2-x^2}}$$

$$= r\int_{\cancel{-r}\ \arcsin(-r/r) = -\frac{\pi}{2}}^{\cancel{r}\ \arcsin\left(\frac{r}{r}\right)=\frac{\pi}{2}} \frac{r\cos\theta}{\sqrt{r^2-r^2\sin^2\theta}}\, d\theta = r\int_{\cancel{-r}\,-\pi/2}^{\cancel{r}\,\pi/2}\frac{\cancel{r\cos\theta}}{\cancel{r\cos\theta}}\, d\theta$$

Trig. sub. $x = r\sin\theta$

$dx = r\cos\theta\, d\theta$

$$= r\int_{\cancel{-r}\,-\pi/2}^{\cancel{r}\,\pi/2} d\theta = r\Big[\theta\Big]_{\cancel{-r}\,-\frac{\pi}{2}}^{\cancel{r}\,\frac{\pi}{2}}$$

$$= r\left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = \pi r.$$

So total circum. of circle $= 2 \cdot \frac{1}{2}$ circum. $= \underline{\underline{2\pi r}}$.

Just as there's an "area up to x function" $\left[\int_a^x f(t)\,dt\right]$

there's an "arc length up to x function"

↳ "arc length function"

$$s(x) = \int_a^x \sqrt{1+(f'(t))^2}\,dt,$$

the length of $y = f(t)$ from $(a, f(a))$ to $(x, f(x))$.

Then by F.T.C. Part I

$$s'(x) = \sqrt{1+(f'(x))^2}.$$

<u>Example</u> Find the arc length function for $y = \ln(\cos x)$ for $0 \le x \le \frac{\pi}{4}$.

<u>Solution</u>

$$s(x) = \int_0^x \sqrt{1+\Big(\underbrace{(\ln(\cos(t)))'}_{\frac{-\sin(t)}{\cos(t)} = -\tan(t)}\Big)^2}\,dt$$

So $\quad s(x) = \displaystyle\int_0^x \sqrt{1 + (-\tan(t))^2}\, dt$

$$= \int_0^x \sqrt{1 + \tan^2(t)}\, dt$$

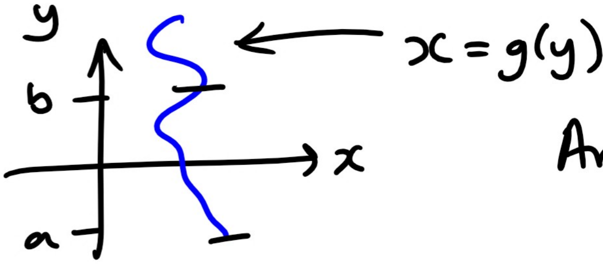
$$= \int_0^x \sec(t)\, dt = \Big[\ln|\sec(t) + \tan(t)|\Big]_0^x$$

$$= \ln|\sec(x) + \tan(x)| - \ln|\cancelto{1}{\sec(0)} + \cancelto{0}{\tan(0)}|$$

$$= \ln|\sec(x) + \tan(x)|. \quad \underbrace{\qquad\qquad}_{0}$$

(No + C ! Definite integral.)

We can deal with curves i.e. $x = g(y)$ (where x is a function of y) just by switching roles (as long as $g'(y)$ continuous).



$$\text{Arc length} = \int_a^b \sqrt{1 + (g'(y))^2}\, dy$$

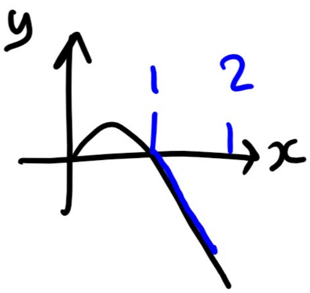
$$= \int_a^b \sqrt{1 + \left(\frac{dx}{dy}\right)^2}\, dy$$

Example Set up the integral for arc length for

$$x = y - \sqrt[3]{y^2} \quad \text{from } y = 1 \text{ to } 2.$$

$y^{2/3}$

Solution



$$\frac{dx}{dy} = 1 - \frac{2}{3} y^{-1/3}$$

$$\text{Arc length} = \int_1^2 \sqrt{1 + \left(1 - \tfrac{2}{3} y^{-1/3}\right)^2}\, dy$$

$\longrightarrow$ Hard.

7.5 Integration Strategy

(Read 7.5 in textbook for examples.)

First KNOW your basic integrals.

(see table p. 503)

Here are some useful guidelines to orient you:

① Simplify! Multiply out brackets, rewrite trig. functions $\left(\tan\theta = \dfrac{\sin\theta}{\cos\theta},\ \sec\theta = \dfrac{1}{\cos\theta}\right)$, etc., cancel.

(2) Substitution

Look for the obvious ones: the $u = g(x)$ where $g'(x)$ is a factor in the integrand. ↑ up to a constant multiple

e.g. $\int x \underbrace{(x^2+5)}_{u = x^2+5}^{3/2}\, dx$

(3) Classify

(a) Trig. integral → follow 7.2 strategy

(b) Rational Function → follow 7.4 strategy

(c) Radicals: $\sqrt{\pm x^2 \pm a^2}$ → Trig. sub. → follow 7.3 strategy

: $\sqrt[n]{x}$ → sub. $u = \sqrt[n]{x}$

(d) Integration by Parts: integrand has form

$u\, dv$

gets simpler when differentiated (u) e.g. x, x^2, $\arcsin(x)$, $\arctan(x)$, $\ln(x)$, …

can integrate (dv) e.g. $\sin(x)$, e^x, … whatever you can integrate.

(4) If at first you don't succeed … try, try again.

(We'll look more at this in the Review session.)