1ZA3 (SECTION C01) - ENGINEERING MATHEMATICS I

Lecture 37

Last time **Integration Strategy**

1. Simplify (multiply out, cancel, rewrite). [5.5]
2. Obvious Substitution ($u = g(x)$ where $g'(x)$ is a factor)
3. Classify
   (a) Trig. integrals ($\cos^3 x \sin^6 x$, $\tan^2 x \sec^5 x$, $\cos 2x \sin 3x$) [7.2]
   (b) Rational functions ($\frac{P(x)}{Q(x)}$) [7.4]
   (c) Radicals → $\sqrt{\pm a^2 \pm x^2}$: trig. substitution [7.3]
   → $\sqrt[n]{x}$: try substitution $u = \sqrt[n]{x}$
   (d) Integration by Parts : $u\,dv$ ← u gets simpler when differentiated; dv can be integrated [7.1]

Integrals & Integration Strategy

1. Inspiration : Problem Sampler (PS) #3 Q8

Find $\displaystyle\int_0^{2\pi} 3^x \cos(x)\,dx = \Big[3^x \sin(x)\Big]_0^{2\pi} - \int_0^{2\pi} (\ln 3) 3^x \sin(x)\,dx$

$\Big[3^x \sin(x)\Big]_0^{2\pi} = 3^{2\pi}\sin(2\pi) - 3^0 \sin(0) = 0$

$$- \ln 3 \int_0^{2\pi} 3^x \sin(x)\,dx$$

$$\frac{d}{dx}(3^x) = \frac{d}{dx}\left(e^{x\ln 3}\right) = (\ln 3)3^x$$

Let $u = 3^x$ $\qquad dv = \cos(x)\,dx$

$\frac{du}{dx} = (\ln 3)3^x \qquad v = \sin(x)$

$t = 3^x$

$dt = (\ln 3)3^x\,dx$

$\frac{ds}{dx} = \sin(x)$

$s = -\cos(x)$

$$= -\ln 3 \left(\left[3^x(-\cos(x)) \right]_0^{2\pi} - \int_0^{2\pi} (\ln 3) 3^x (-\cos(x))\, dx \right)$$

$$= -\ln 3 \left(\underbrace{\left(3^{2\pi}(-1) - 3^0(-1)\right)}_{-3^{2\pi}+1} + \ln 3 \int_0^{2\pi} 3^x \cos(x)\, dx \right)$$

Call

$A = \int_0^{2\pi} 3^x \cos(x)\, dx$. We've just shown:

$$A = -\ln 3 \left(-3^{2\pi} + 1 + (\ln 3) A \right)$$

Solve for A: $\quad A\left(1 + (\ln 3)^2\right) = \ln 3 \left(3^{2\pi} - 1\right)$

$$\Rightarrow A = \frac{(\ln 3)\left(3^{2\pi} - 1\right)}{1 + (\ln 3)^2}.$$

Take-home message: using the cyclical nature of derivatives of trig. functions together with integration by parts can give an equation to solve for your original integral.

② Inspiration: various, including PS #3 Q16

Find $\displaystyle\int \frac{dx}{x\sqrt{9+x^2}} = \int \frac{3\sec^2\theta\, d\theta}{3\tan\theta \sqrt{\underbrace{9 + 9\tan^2\theta}_{9\sec^2\theta}}}$

$x = 3\tan\theta$

$dx = 3\sec^2\theta\, d\theta$

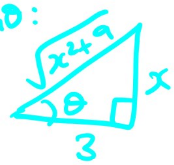
Take-home message: Check your table on p. 503!!

$$= \int \frac{\cancel{3}\sec^{\cancel{2}}\theta}{\cancel{3}\tan\theta \cdot 3\cancel{\sec\theta}}\, d\theta = \frac{1}{3}\int \frac{1}{\cancel{\cos\theta}} \cdot \frac{\cancel{\cos\theta}}{\sin\theta}\, d\theta$$

$\int \csc^2\theta\, d\theta$

$= -\cot\theta\, d\theta$

$\frac{x}{3} = \tan\theta$:



$$= \frac{1}{3}\int \csc\theta\, d\theta = \frac{1}{3}\ln|\csc\theta - \cot\theta| + C$$

$$= \frac{1}{3}\ln\left| \frac{\sqrt{x^2+9}}{x} - \frac{3}{x} \right| + C_1 = \frac{1}{3}\left(\ln\left|\sqrt{x^2+9} - 3\right| - \ln|x|\right) + C.$$

(3) Inspiration : PS # Q21

Find $\displaystyle\int \underbrace{\ln(x^2+5)}\, dx$

$x = \sqrt{5}\tan\theta$

$dx = \sqrt{5}\sec^2\theta\, d\theta$?

Take-home: Have faith in the strategy but here are some tips on how to be smart about using it:

$u = \ln(x^2+5) \quad dv = dx$

$\dfrac{du}{dx} = \dfrac{2x}{x^2+5} \qquad v = x$

$$= x\ln(x^2+5) - \underbrace{\int \frac{2x^2}{x^2+5}\, dx}$$

$$\int \frac{2x^2}{x^2+5}\, dx = \int 2 - \underbrace{\frac{10}{x^2+5}}\, dx$$

Now try that trig. sub.

$x = \sqrt{5}\tan\theta$ ...

$$\int \frac{10}{x^2+5}\, dx = \int \frac{\cancel{10}^{\,2}\sqrt{5}\,\cancel{\sec^2\theta}\, d\theta}{\cancel{5}\,\cancel{\sec^2\theta}}$$

$x = \sqrt{5}\tan\theta$

$dx = \sqrt{5}\sec^2\theta\, d\theta$

$= 2\sqrt{5}\int d\theta = 2\sqrt{5}\,\theta = 2\sqrt{5}\arctan\left(\frac{x}{\sqrt{5}}\right) + C.$

Be smarter:

(1) Use trig. identities, complete the square, rationalize a denominator — ways to make up what you've got

(2) Force a substitution — think Case III & IV from $P(x)/Q(x)$ (7.4)

(3) Try int. by parts with just one function (e.g. $\sin^{-1}(x)$)

(4) Don't be afraid to mix techniques take several steps.

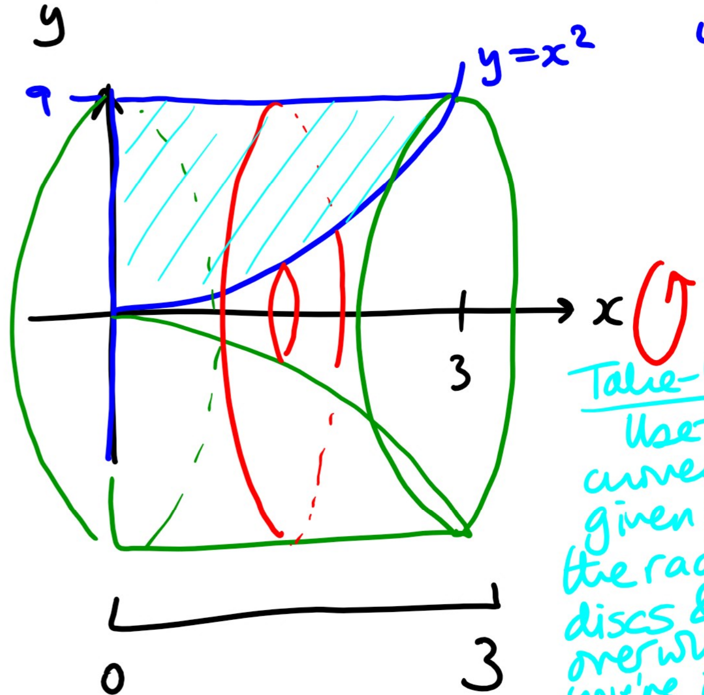
$$\begin{array}{r|l} & 2 \leftarrow S(x) \\ x^2+5 & 2x^2 \\ & 2x^2+10 \\ \hline & -10 \leftarrow R(x) \end{array}$$

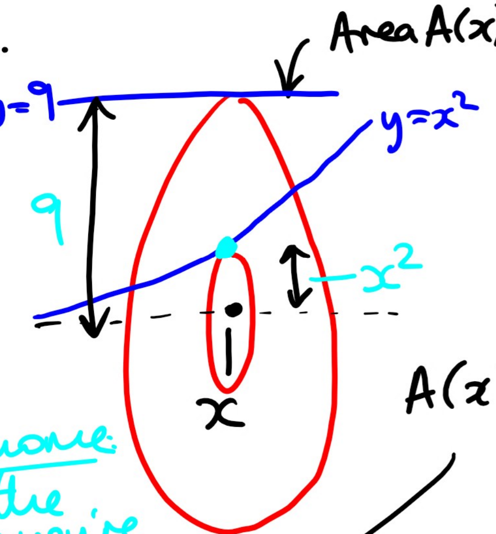
SO ANSWER $= x\ln(x^2+5) - 2x + 2\sqrt{5}\arctan(x/\sqrt{5}) + C$.

Volumes of Solids of Revolution

Inspiration: PS #3 Q5

(5) Find the volume obtained by rotating the region bounded by y-axis, $y = x^2$ & $y = 9$ around the x-axis.





Volume $= \int_0^3 A(x)\,dx$

$A(x)$ = area of outer disc − area of inner disc

Take-home: Use the curves you're given to tell you the radii of the discs & the region over which you're integrating.

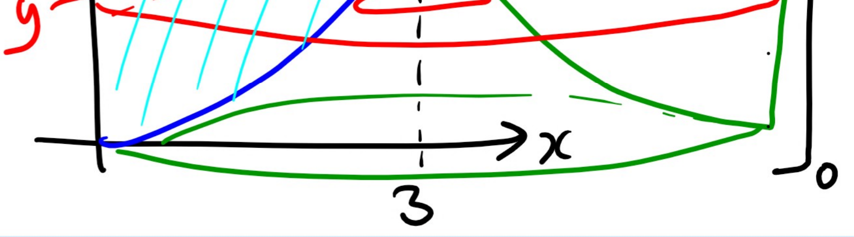
$= \pi 9^2 - \pi (x^2)^2 = 81\pi - \pi x^4$

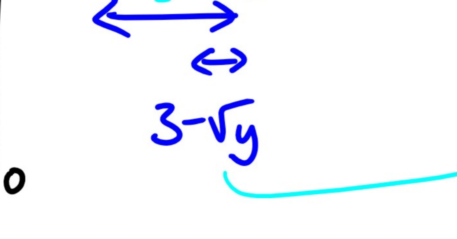
Volume $= \int_0^3 81\pi - \pi x^4 \, dx$

$= \left[81\pi x - \frac{\pi x^5}{5} \right]_0^3 = \ldots =$ $243\pi - \frac{243\pi}{5} - 0$

$\underline{\underline{194.4\pi}}$

Now rotate the same region around $x = 3$. Set up the integral.





Area = area of outer disc − area of inner disc

$= \pi 3^2$

$- \pi(3-\sqrt{y})^2$

Integral: $$\int_0^9 9\pi - \pi(3-\sqrt{y})^2\, dy.$$

$$= \int_0^9 9\pi - \pi(9 - 6\sqrt{y} + y)\, dy$$

$$= \int_0^9 \cancel{9\pi} - \cancel{9\pi} + 6\pi\sqrt{y} - \pi y\, dy$$

$$= \pi\int_0^9 6\sqrt{y} - y\, dy.$$

FINALLY: (1) Course Evaluations: please take a 10-minute break before Thursday to review this course & help make it better (or tell us what we're doing right!).
→ see link on website.

(2) It may not be much time until the exam but you've got this — think like our integration strategy & break every problem into bite-size chunks. Do whatever you can do — it's all anyone can ever ask!

GOOD LUCK!!! & thanks for all the fish...