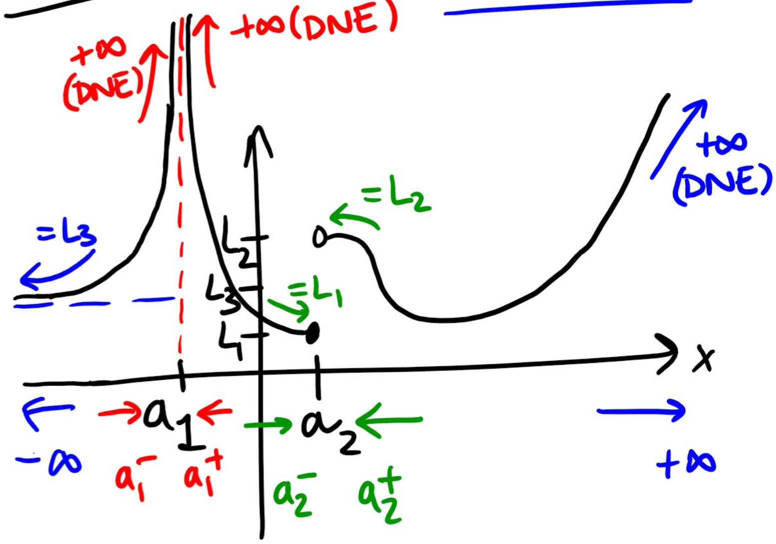
1ZA3 (SECTION C01) – ENGINEERING MATHEMATICS I

Lecture 6

Last time LIMITS

- limits at a point
- infinite limits
- limits at infinity



Remember: "limit exists" means it equals a NUMBER.

If a limit is $\pm\infty$, then it does not exist (DNE).

Continuity A function $f(x)$ is continuous at the point a if $\lim_{x\to a} f(x) = f(a)$

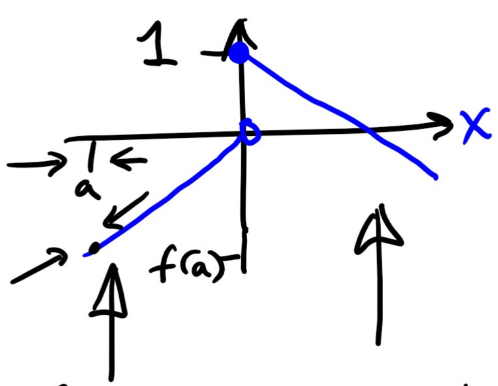
This says
- $\to$ f is defined at a
- $\to$ $\lim_{x\to a} f(x)$ exists

as well as telling us these two #s are equal

(We arrive at $f(a)$ as x "tends to" a.)

Example

$$f(x) = \begin{cases} x & x<0 \\ 1-x & x\geq 0 \end{cases}$$



f continuous at $x \neq 0$

$$\lim_{x\to 0^+} f(x) = f(0) = 1 \quad (*)$$

$$\lim_{x\to 0^-} f(x) = 0 \neq f(0)$$

f NOT continuous at 0

(Jump discontinuity.)

So $\lim_{x \to 0} f(x)$ DNE

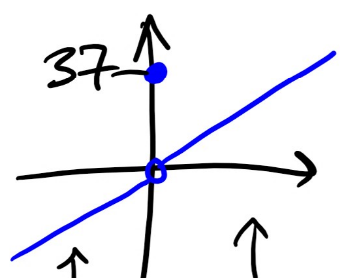
limit from right at 0 (0^+) is equal to $f(0)$

Because of (*) we say f is <u>right continuous at 0</u>

(& it is <u>not</u> left continuous at 0) since $\lim_{x \to 0^-} f(x) \neq f(0)$.

$$\boxed{\begin{array}{l} \text{If } \lim_{x \to a^+} f(x) = f(a) \text{ then } f \text{ is right continuous} \\ \quad \lim_{x \to a^-} f(x) = f(a) \text{ then } f \text{ is left continuous.} \end{array}}$$

<u>Example</u> $f(x) = \begin{cases} x & x \neq 0 \\ 37 & x = 0 \end{cases}$



continuous for $x \neq 0$

$\lim_{x \to 0^-} f(x) = 0 = \lim_{x \to 0^+} f(x)$

$\neq 37 \qquad \neq 37$

f is not continuous OR left continuous OR right continuous at $x = 0$

We can think of the discontinuity at $x=0$ being "removed" if we just change <u>one value</u> of $f(x)$ $x=0$ i.e. redefine $f(0)$ so $f(0)=0$ & then work with new f instead

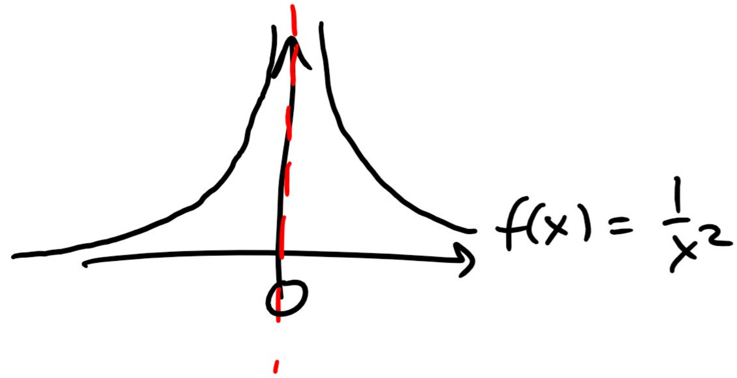
(Here $x=0$ is a REMOVEABLE DISCONTINUITY)

i.e. the limit at 0 does exist !!!

Notice here $\lim_{x \to 0} f(x) = 0$ (as left & right limits at 0 both exist) but NOT cont. at 0.

$(\neq f(0))$

Example $f(x) = \frac{1}{x^2}$



At $x = 0$:

$\lim_{x \to 0^-} \frac{1}{x^2} = \infty$ DNE

$\lim_{x \to 0^+} \frac{1}{x^2} = \infty$ DNE

So $\lim_{x \to 0} \frac{1}{x^2}$ DNE so $\frac{1}{x^2}$ not continuous at $x=0$

INFINITE DISCONTINUITY

A function $f(x)$ is continuous on an interval (a,b) or $[a,b]$ or $[a,b)$ or $(a,b]$ or $(-\infty,b]$ or $(-\infty,b)$ or $[a,\infty)$ or (a,∞) or $(-\infty,\infty)$ if it is continuous at every point in the interval.

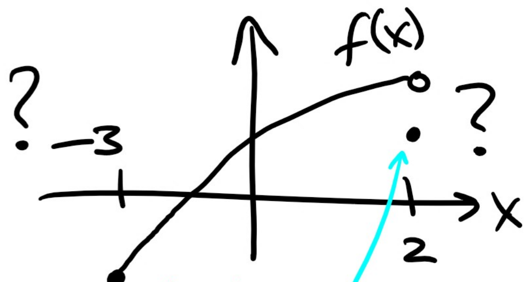
"finite" intervals

"infinite" intervals

When we say "an interval" without further info. we mean it could be any of these things!!!

We include right endpoint in interval if $f(x)$ is left continuous there & the left endpoint if $f(x)$ is right continuous there.

Example



Even if $f(2)$ is defined to be something here in this setup, if that value is not $\lim_{x \to 2^-} f(x)$, then $f(x)$ not left continuous at 2, so 2 not in the interval

f continuous everywhere between -3 and 2

f continuous from right at $x = -3$

f is NOT continuous from left at $x = 2$

So we can (only) say that f is continuous on $[-3, 2)$.

Examples of Functions that are continuous

Please do not forget this part!!! Just because a function is on this list does not necessarily mean it is continuous everywhere!! → **Where they are defined**

→ polynomials e.g. x^2+6x-3, $5x^{72}+6x^{13}$

→ rational functions e.g. $\dfrac{x^2+7}{x-5}$ ← Not defined at $x=5$ but continuous for $x \neq 5$

→ root functions e.g. $\sqrt[6]{x}$ ← $(0,\infty)$ for x even; $(-\infty,\infty)$ for x odd

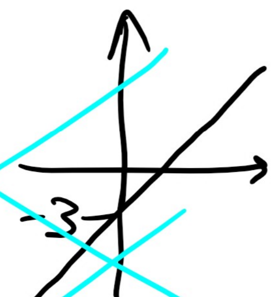
→ trig. functions → inverse trig. functions

→ exponential functions → log functions

Example Where is $\dfrac{x^2-x-6}{x+2}$ continuous?

Solution

$$= \frac{(x+2)(x-3)}{x+2} = x-3$$



This is all to do with the subtle issue of "removable discontinuities"! See below.

This is just another way of writing

$$y = x-3$$

So be careful!! Cancel things out so you can see what's really going on!

i.e. this function ~~is~~ can be thought of as continuous everywhere!!

This is a subtle and important aspect of how everything is set up, and we will follow the textbook approach to be sure we are consistent. The good news is that for most people this seems consistent with information coming from high school. We will discuss this in further detail in class, but, for this example, what is going on here is as follows:

<u>Q</u> Where is $f(x) = \dfrac{x^2 - x - 6}{x+2}$ continuous?

<u>A</u>

- $f(x)$ is not defined at $x = -2$ so certainly $f(x)$ cannot be continuous at $x = -2$.
- Everywhere else i.e. for $a \neq -2$, $f(a) = \lim_{x \to a} f(x)$ so f <u>is</u> continuous at all other a.

We also call $x = -2$ a <u>removeable discontinuity</u> of $f(x)$ even though $f(x)$ is <u>not</u> even defined at $x = -2$

because we can decide on a way to define a value of $f(-2)$ that makes the discontinuity problem go away. We end up doing this in the "obvious" way, but there are 2 ways you should think about it:

1. We "know", intuitively, that for all other values of x $(\neq -2)$, that $f(x) = x-3$, so we define $f(-2) = -2-3 = -5$.

This is the approach that I was giving a brief version of above. Since, however, we are not always so lucky as to know how to 'cancel' in this way, then we will typically do the following to force continuity at the 'removable discontinuity':

2. We want in this case to have continuity at $x=-2$.

So decide $f(-2) = \lim_{x \to 2} f(x)$!!! [This is the definition of continuity at -2 !!]

Then $f(-2) = \lim_{x \to -2} \dfrac{x^2-x-6}{x+2} = ???$ ... Well we can't use limit laws, since $\lim_{x \to -2} x+2 = 0$.

But $f(x) = \dfrac{x^2 - x - 6}{x+2} = \dfrac{(x+2)(x-3)}{x+2}$ is defined

for $x \neq -2$, and remember that when we talk about limits we talk about the values $f(x)$ takes near the point of interest (here that point is -2). At points x near -2, we can say

that $f(x) = \dfrac{\cancel{(x+2)}(x-3)}{\cancel{x+2}} = x - 3.$

allowed as the x-values we look at with this limit are the x near -2

So in terms of limits: $\lim\limits_{x \to -2} f(x) \overset{=}{=} \lim\limits_{x \to -2} (x-3)$

$= -5.$

But notice, this was again very nice because we could cancel. What if you can't cancel? We will look in a few weeks at other methods of finding limits of the form $\lim\limits_{x \to a} \dfrac{f(x)}{g(x)}$ where $f(a) = 0 = g(a)$!!!