1ZA3 (SECTION C01) – ENGINEERING MATHEMATICS I

Lecture 7

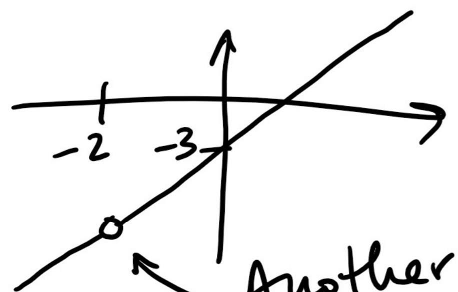
Last time

A vision of the future ...

... when we will be able to remove "removable discontinuities"!

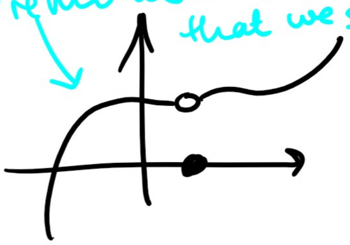
e.g. work with $x-3$ instead of $\dfrac{(x+2)(x-3)}{x+2}$.

BUT NOT YET!!!...



Another example of a removable discontinuity

This was the type of "removable discontinuity" that we saw last lecture.



But what about:

$$f(x) = \frac{x^{92} - 10x^{76} + 5x^{52} - 50x^{36} + 7x^{16} - 70}{x^{28} + 2x^{18} - 10x^{12} - 20x} \;?$$

← Here there is a common factor, but finding it is unpleasant

$$f(x) = \frac{\ln(x)}{x-1} \;?$$

What about $x=1$? Should know if we can stop worrying about that point.

If we have two functions $f(x)$, $g(x)$, both continuous at $x=a$, then so are

$f(x) \pm g(x)$, $f(x)g(x)$, $f(x)/g(x)$ as long as $g(x) \neq 0$

Compositions

$f(x)$, $g(x)$ with $f(x)$ continuous at $b = \lim_{x \to a} g(x)$ ← So this says that the limit exists and is some # called b.

Then $\lim_{x \to a} f(g(x)) = f(b) \quad \left(= f\left(\lim_{x \to a} g(x) \right) \right)$

$f \circ g$

i.e. can swap limit & the application of f. IF f cont. at the special value b as defined above.

In particular if $b = g(a)$ i.e. g cont. at a,

then $\lim_{x \to a} (f(g(x))) = f(g(a))$

"i.e. a cont. funct. of a cont. funct. is cont."

Example When is $\ln(1 + \sin(x))$ continuous?

Solution $\ln$ is continuous where defined i.e. for inputs > 0

So where is $1 + \sin(x) > 0$

i.e. $\sin(x) > -1$ i.e. NOT when $\sin(x) = -1$

($\sin(x)$ takes values in $[-1, 1]$)

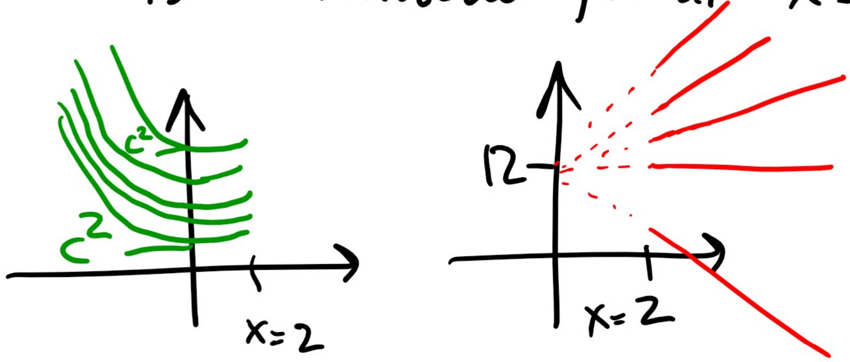
i.e. NOT $x = \frac{3\pi}{2}$ $\cancel{\frac{(2n-1)\pi}{2}}$, $\frac{7\pi}{2}$, ..., $-\frac{\pi}{2}$, $-\frac{5\pi}{2}$, ...

↑ This was a guess at the general formula that was not quite correct. General formula: $x = \frac{4n-1}{2}\pi$.

Example Find c so that $f(x) = \begin{cases} x^2 + c^2, & x < 2 \\ cx + 12, & x \geq 2 \end{cases}$

is continuous for all x.

Solution



Here we need

$$\begin{cases} x^2 + c^2 = cx + 12 \\ \text{at } x = 2 \end{cases}$$

(i.e. left & right limits equal) at $x = 2$

i.e. $4 + c^2 = 2c + 12$

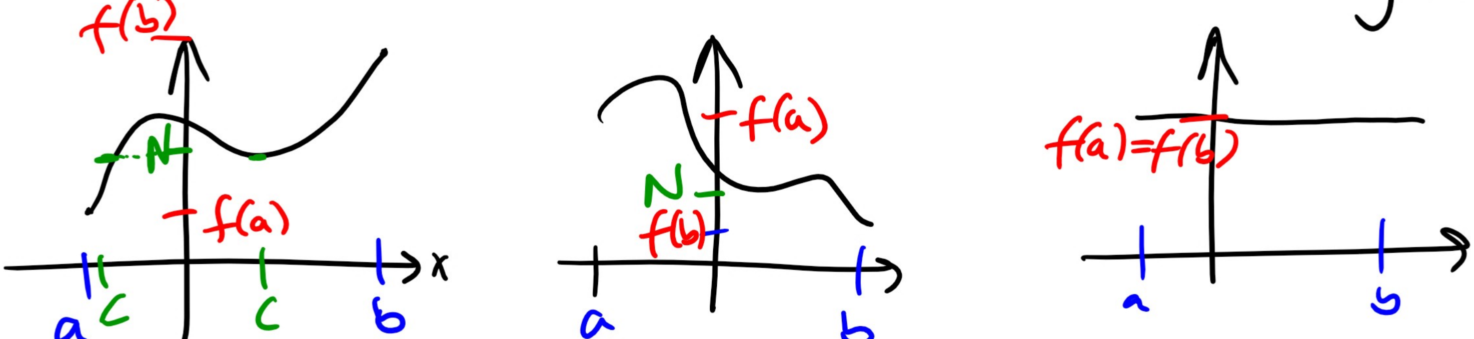
$c^2 - 2c - 8 = 0$

$(c+2)(c-4) = 0$ so $c = -2$ or $c = 4$.

Yes both are solutions: try drawing the graphs.

Intermediate Value Theorem

↳ Continuous functions can't jump.



IVT If f is continuous on $[a,b]$ and N is any intermediate value between $f(a)$ and $f(b)$ (inclusive), then there is some x-value c in $[a,b]$ with $f(c) = N$.

Example: Use IVT to show that

$$f(x) = e^{\sin\left(\frac{\pi x}{2}\right)} - x$$

has a root in $[1,2]$.

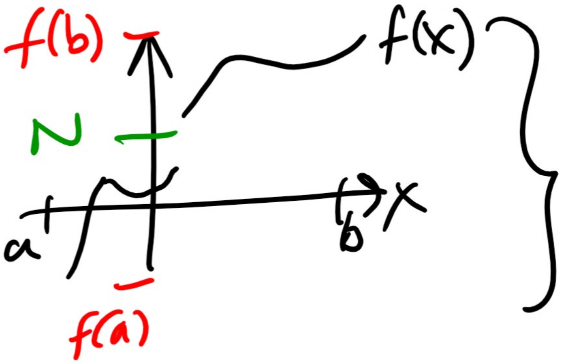
Solution: To use IVT we want to take $N=0$ BUT need to know 0 between $f(1)$ and $f(2)$.

(As discussed in class, you should also check that $f(x)$ is cont. on $[1,2]$.)

$$f(1) = e^{\sin(\pi/2)} - 1 = e - 1 \approx 1.7 > 0$$

$$f(2) = e^{\sin(\pi)} - 2 = -1 < 0.$$

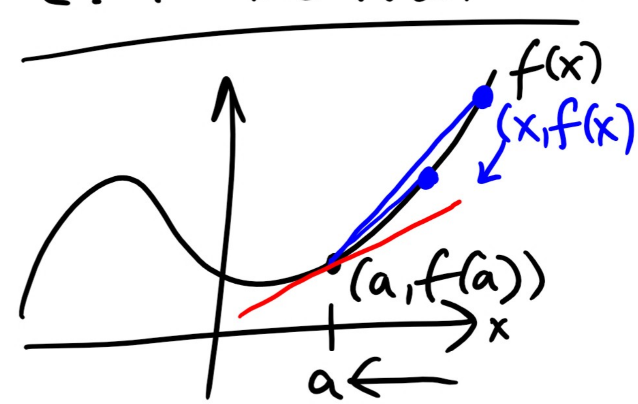
Since 0 lies between $f(1)$ and $f(2)$, by IVT there is some c in $[1,2]$ with $f(c)=0$. (i.e. a root).



Need to know f continuous to use IVT.

← Here $f(x)$ not continuous & "jumps" over the intermediate value N.

2.7 Derivatives



The tangent line to $y=f(x)$ at the point $(a,f(a))$ is the line through $(a,f(a))$ with slope:

$$m = \lim_{x \to a} \frac{f(x)-f(a)}{x-a}$$

Or equivalently $m = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$

in both cases we have 2

i.e. Both one-sided limits exist.