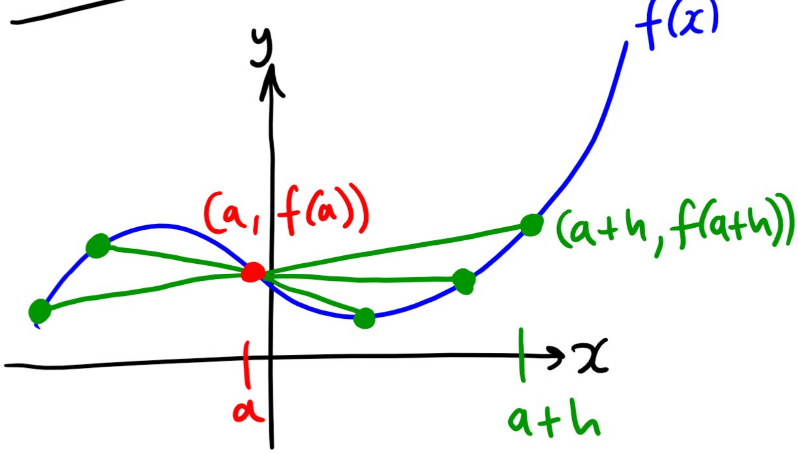
1ZA3 (SECTION C01) – ENGINEERING MATHEMATICS I

Lecture 8

Last time



The tangent line to $y = f(x)$ at $(a, f(a))$ is the line through $(a, f(a))$ with slope:

$$m = \lim_{h \to 0} \left(\frac{f(a+h) - f(a)}{h} \right).$$

Example Find the equation of the tangent line to $f(x) = \frac{5}{x}$ at the point $(5, 1)$.

Sketch of Solution Tangent line has form $y = mx + c$

Find $m = \lim_{h \to 0} \frac{f(5+h) - f(5)}{h} = \lim_{h \to 0} \left(\frac{\frac{5}{5+h} - 1}{h} \right)$

$$= \lim_{h \to 0} \left(\frac{5 - (5+h)}{h(5+h)} \right) = \underset{\text{gap!}}{\overset{\text{Fill in the}}{\cdots}} = \lim_{h \to 0} \left(\frac{-1}{5+h} \right) = -\frac{1}{5}$$

Now $(5, 1)$ is a solution to $y = -\frac{1}{5}x + c$

i.e. $1 = -\frac{1}{5} \cdot 5 + c \quad \underset{\text{Fill in the gap!}}{\cdots} \Rightarrow c = 2.$

So answer is $y = -\frac{1}{5}x + 2.$

The derivative of a function $f(x)$ at a number a is the slope of the tangent line to $y=f(x)$ at $(a, f(a))$ denoted $f'(a) = \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\right)$.

In Physics, if $f(x)$ is the position function at time x, the slope $f'(a)$ is the velocity at time $x=a$.

In general $f'(a)$ is the instantaneous rate of change of $f(x)$ at $x=a$.

Example Find $f'(a)$ if $f(x) = 2x^2 - 3x + 7$.

Solution
$$f'(a) = \lim_{h\to 0}\left(\frac{f(a+h)-f(a)}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{2(a+h)^2 - 3(a+h) + 7 - (2a^2-3a+7)}{h}\right)$$
$$= \lim_{h\to 0}\left(\frac{\cancel{2a^2} + 4a\cancel{h} + 2h^{\cancel{2}} \cancel{-3a} - 3\cancel{h} \cancel{+7} \cancel{-2a^2} \cancel{+3a} \cancel{-7}}{\cancel{h}}\right)$$
$$= \lim_{h\to 0}(4a + \underbrace{2h}_{\to 0} - 3) = 4a - 3.$$

Example Consider $\lim_{h\to 0}\left(\frac{\sqrt[5]{32+h} - 2}{h}\right)$. For which $f(x)$ and # a is this $f'(a)$?

match up terms.

Solution We know $f'(a) = \lim_{h\to 0} \left(\dfrac{f(a+h) - f(a)}{h} \right)$

So can we find $f(x)$ and a with

$$f(a+h) = \sqrt[5]{32+h} \quad \text{and} \quad f(a) = 2 \quad ?$$

We see $f(x) = \sqrt[5]{x}$ and $a = 32$.

Example Sketch the graph of $y=f(x)$ if

$f(0) = f(2) = f(3) = 0$ and

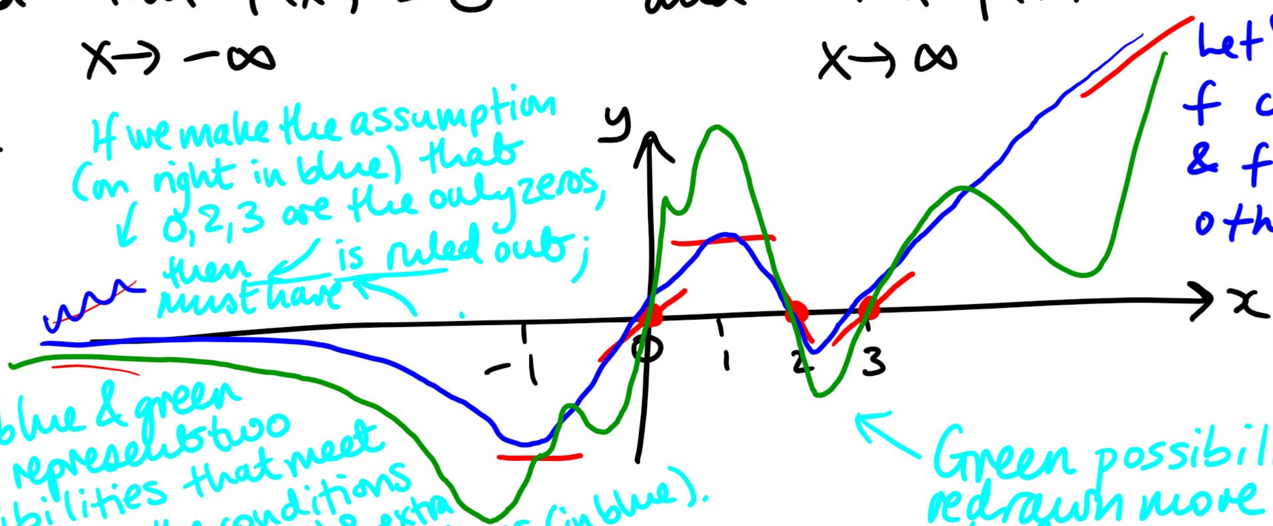
$f'(-1) = f'(1) = 0$ and $f'(0) = f'(3) = 1$ and $f'(2) = -1$

and $\lim_{x\to -\infty} f(x) = 0$ and $\lim_{x\to\infty} f(x) = \infty$

Soln

Let's assume f continuous & f has no other roots

We do this just to help ourselves understand — if you are not told these things are true, you cannot assume them!!



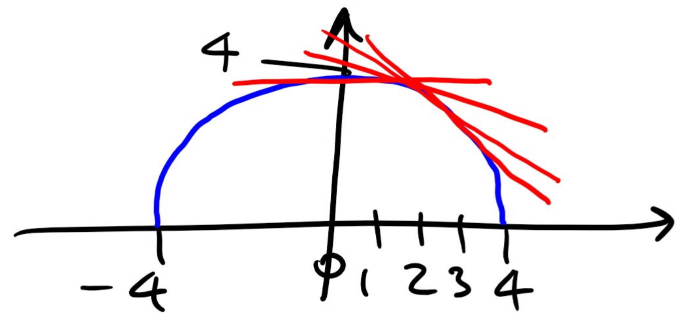
2.8 Derivative as a function

Notice that $f'(x) = \lim_{h\to 0} \left(\dfrac{f(x+h) - f(x)}{h} \right)$

is a function of x.

(Now treat "a" as a variable "x"!)

Example $f(x) = \sqrt{16-x^2}$



$f'(0) = 0$

$f'(1) \sim -\frac{1}{4}$ (say)

$f'(2) \sim -\frac{1}{2}$ (say)

$f'(3) \sim -1$ (say)

What about at $x = 4$?

Or $f'(x)$ in general? ... TBC

So as x changes, the value of $f(x)$ changes!

Definition We say that a function $f(x)$ is **differentiable** at a point a if $f'(a)$ exists.

And $f(x)$ is differentiable on (a,b) (which could have $a = -\infty$ or $b = \infty$) if $f(x)$ is differentiable at every point in the interval.

Note: endpoints not included.

Examples $f(x) = \sqrt[3]{x}$ and $f(x) = |x-3|$.

Where are these functions differentiable?

TBC next time...