1ZA3 (SECTION C01) — ENGINEERING MATHEMATICS I

Lecture 9

Last time Differentiability

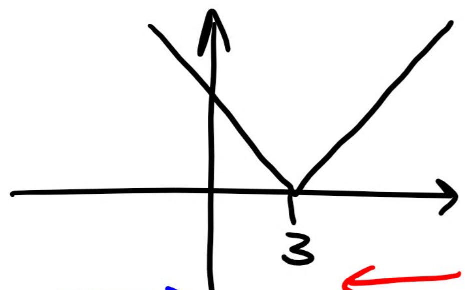
Differentiating a function $f(x)$ = finding its derivative $f'(x)$.

A function $f(x)$ is differentiable at a if $f'(a)$ exists

and $f(x)$ is differentiable on (a,b) if $f'(c)$ for every $c \in (a,b)$.

3 reasons why functions fail to be differentiable

Example $f(x) = |x-3|$



"$f'(3)$" does not exist

This means:

$$\lim_{h\to 0} \frac{f(3+h)-f(3)}{h} \text{ DNE}$$

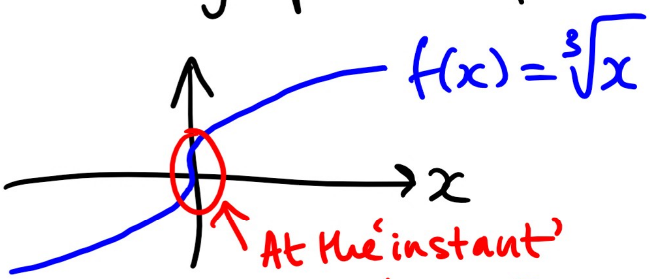
$$\lim_{h\to 0^-} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$$

$$\lim_{h\to 0^+} \frac{f(3+h)-f(3)}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$$

(1) "Corner"/"Kink" — left & right limits do not agree

Example $f(x) = \sqrt[3]{x}$

Look at graph: reflect graph of $y = x^3$ in line $y = x$:



It can be shown that

$$f'(x) = \frac{1}{3(\sqrt[3]{x})^2}, \quad x \neq 0$$

$= \frac{1}{3}x^{-\frac{2}{3}}$

$$\lim_{x \to 0} f'(x) = \frac{1}{\text{small +ve}} = \infty$$

② "Vertical Tangent". ← Limit is infinite

Notice $f(x) = |x-3|$ and $f(x) = \sqrt[3]{x}$ are both continuous i.e. continuity does not guarantee differentiability. But fortunately we <u>can</u> go the other way:

<u>Theorem</u> If $f(x)$ is differentiable at $x = a$, then $f(x)$ is continuous at $x = a$.

(See textbook pp. 156-157)

③ So this tells us that at any discontinuity of $f(x)$, say at $x = a$, $f(x)$ is NOT differentiable at $x = a$.

<u>Notation</u> Sometimes $f'(x)$ is written

i.e. how a dependent variable y depends on (indep. var.) x.

not a ratio! → $\frac{df}{dx} = \frac{d}{dx} f(x)$ & if interested in dependency $y = f(x)$ also write $\frac{dy}{dx}$ or y'

The rate of change <u>of y</u>.

Higher Derivatives

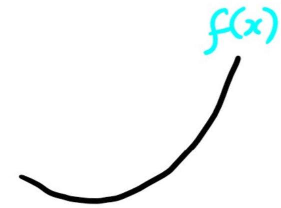
$f'(x)$ is a function (where defined) so can define its derivative written $f''(x) = \left(f'(x)\right)'$

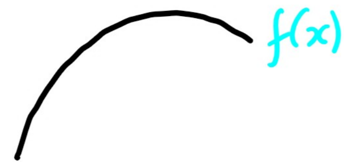
(Also $\dfrac{d^2 f}{dx^2} = \dfrac{d^2}{dx^2} f(x) = y'' = \dfrac{d^2 y}{dx^2}$.)

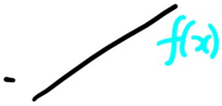
→ called 2nd derivative

$f''(x)$ is the rate of change of $f'(x)$ at x

i.e. $f''(x)$ tells us how slope of tangent lines to graph of $f(x)$ are changing:

So $f''(x) > 0$: graph of $f(x)$ getting steeper e.g. 

$f''(x) < 0$: graph of $f(x)$ getting less steep e.g. 

$f''(x) = 0$: slope stays same e.g. 

& so on ... for any non-negative integer n we can define

$$f^{(n)}(x) = \frac{d^n f}{dx^n} = \ldots = \frac{d^n}{dx^n} f(x) = y^{(n)} = \frac{d^n y}{dx^n}$$

"nth derivative".

Chapter 3 Derivatives of Polynomials, Exp. Functions, Trig. Functions, Product Rule.

Useful rules

If $f(x)$, $g(x)$ differentiable at $x=a$ then so are

$f(x) \pm g(x)$, $cf(x)$, $f(x)g(x)$ and moreover:

constant c

$$(f(x) \pm g(x))' = f'(x) \pm g'(x)$$ Sum / Difference Rules

$$(cf(x))' = cf'(x)$$ Constant Multiple Rule

$$(f(x)g(x))' = f'(x)g(x) + f(x)g'(x)$$ Product Rule (or Leibniz Rule)

Also $\dfrac{f(x)}{g(x)}$ is differentiable if $g(x) \neq 0$

if $f(x)$ & $g(x)$ are differentiable too, as above

and $$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x)g(x) - f(x)g'(x)}{(g(x))^2}$$ Quotient Rule

Polynomials

The simplest polynomial

$y = c$ (constant!)

$y' = 0$ (zero slope)

Next simplest: $y = x$

$$y' = 1 = \lim_{h \to 0} \frac{(x+h)-x}{h} = \lim_{h \to 0} (1)$$

Now $y = x^2$: $y' = (x^2)' = (x.x)'$ Product Rule

$= 1.x + x.1$

$= x + x = 2x$

Now $y = x^3$: $y' = (x.x^2)'$

$= 1.x^2 + x(2x) = x^2 + 2x^2$

$= 3x^2$

$$\boxed{\text{If } f(x) = x^n \text{, then } f'(x) = n x^{n-1}}$$

Power Rule.

[See also two other explanations for where this rule comes from in the textbook. The argument above is not rigorous — how do we know that if we keep going, at the 28th step we output $28x^{27}$? We'll be able to justify this in Semester 2 with something called the "Principle of Mathematical Induction."]