1B03 – LINEAR ALGEBRA 1 (C01) WS19 Lecture 11

Last Time Determinants by Cofactor Expansion

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & & & & & \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & & & & \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

- A – $n \times n$ matrix.
- Pick a row or column of A
- Go along the chosen row (or go down the chosen column) multiplying entries by their Cofactors : $C_{i,j} = (-1)^{i+j} \underbrace{M_{i,j}}$ ← Minor of a_{ij} $= \det(A[i;j])$.
- Add up the resulting products $= \det(A)$.

Useful trick for working out $(-1)^{i+j}$:

They form a chequerboard :

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

etc.

Useful insight : $a_{ij} C_{i,j} = 0$ if $a_{ij} = 0$ so

Pick a row/column with lots of zeros!

__Example__ $A = \begin{bmatrix} 3 & 1 & 0 & 5 \\ 6 & 7 & 2 & 3 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$. Find $\det(A)$.

Notice: row 4 & column 3 have lots of zeros, so pick one to expand along/down (doesn't matter which) — we'll choose column 3

$$\det(A) = +0 - 2\begin{vmatrix} 3 & 1 & 5 \\ 0 & -2 & 2 \\ 0 & 0 & -1 \end{vmatrix} + 0 - 0$$

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

$$= -2\left(3\begin{vmatrix} -2 & 2 \\ 0 & -1 \end{vmatrix}\right) = -2\left(3(-2)(-1) - \cancel{2(0)}\right)$$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$$= -12.$$

Notice that $\begin{vmatrix} 3 & 1 & 5 \\ 0 & -2 & 2 \\ 0 & 0 & -1 \end{vmatrix} = 3(-2)(-1)$

__Fact__ The determinant of __any__ upper triangular, lower triangular (or diagonal) matrix is $a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}$ $\left(\begin{bmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{bmatrix}\right)$.

<u>Remember</u> Matrix A, U.T. or L.T. or diagonal is invertible iff all diagonal entries $a_{ii} \neq 0$

(if and only if)

Notice this is the same as saying $\det(A) \neq 0$.

(Consistent with $\det(A) \neq 0 \iff A$ invertible which we will justify in general later.)

A neat trick for 3×3 matrices ONLY.

<u>Example</u> $A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & -2 & 3 \\ -3 & 1 & 0 \end{bmatrix}$. Find $\det(A)$.

<u>Solution</u>

$$\begin{array}{ccccc} c_1 & c_2 & c_3 & c_1 & c_2 \\ 3 & 2 & -1 & 3 & 2 \\ 0 & -2 & 3 & 0 & -2 \\ -3 & 1 & 0 & -3 & 1 \end{array}$$

$$\det(A) = 3(-2)(0) + 2(3)(-3) + (-1)(0)(1)$$
$$- (-1)(-2)(-3) - (3)(3)(1) - 2(0)(0)$$
$$= 0 - 18 + 0 + 6 - 9 - 0 = -$$

2.2 Evaluating determinants by Row Reduction

E.R.Os

Suppose $A \xrightarrow{\text{E.R.O.}} B$. What happens to $\det(A)$?

Depends on E.R.O. :

E.R.O.	det(B)
(1) Scale a row of A by $k \neq 0$. $\longrightarrow$	$\det(B) = k\det(A)$.
(2) Add a scalar multiple of row of A to another row of A. $\longrightarrow$	$\det(B) = \det(A)$.
(3) Swap over any two rows of A $\longrightarrow$	$\det(B) = -\det(A)$

Look at special case $A = I$, $B = E$ (elementary).

$\det(I) = 1$ so in case (1) $\det(E) = k$

(2) $\det(E) = 1$

(3) $\det(E) = -1$

For each type of E.R.O.

$A \xrightarrow[\text{\& elem. matrix} = E]{\text{E.R.O.}} B = EA$ so $\det(B) = \det(E)\det(A)$

Notice $\det(\text{el. matrix}) \neq 0$.

Notice also : if $\det(A) = 0$ then after ERO det. stays 0.