1B03 – LINEAR ALGEBRA 1 (C01) WS19 Lecture 12

Yesterday

- Deteminants of triangular & diagonal matrices are easy: $\det(A) = a_{11} a_{22} \cdot \ldots \cdot a_{nn}$.

- If $A \xrightarrow[\text{with elementary matrix } E]{\text{E.R.O.}} B = EA$, then $\det(B) = \det(E) . \det(A)$,

$$\det(E) = \begin{cases} k & \text{if E.R.O. scales a row of } A \text{ by } k \neq 0 \\ 1 & \text{if E.R.O. adds a row of } A \text{ to another row of } A \\ -1 & \text{if E.R.O. swaps over 2 rows of } A. \end{cases}$$

Goal: Turn matrix A into triangular matrix using EROs

easy det to compute

Keep track of how EROs change det so we can recover $\det(A)$.

Also on lookout for row/column of zeros $\Rightarrow \det(A) = 0$

Example Find $\begin{vmatrix} 1 & 5 & 3 \\ 2 & -1 & 2 \\ -1 & 0 & 6 \end{vmatrix}$ by Row Reduction.

Solution

If in doubt, use GJ Elim. rules.

$$= \begin{vmatrix} 1 & 5 & 3 \\ 0 & -11 & -4 \\ 0 & 5 & 9 \end{vmatrix} = -11 \begin{vmatrix} 1 & 5 & 3 \\ 0 & 1 & 4/11 \\ 0 & 5 & 9 \end{vmatrix} = -11 \begin{vmatrix} 1 & 5 & 3 \\ 0 & 1 & 4/11 \\ 0 & 0 & 9-\frac{20}{11} \end{vmatrix}$$

$$\det(RHS) = -\frac{1}{11}\det(LHS) \qquad = \frac{79}{11}$$

$$\det(B) = k \cdot \det(A)$$

$$\Rightarrow \det(LHS) = -11\det(RHS)$$

$$\det(A) = \frac{1}{k}\det(B).$$

$$= -11\begin{vmatrix} 1 & 5 & 3 \\ 0 & 1 & 4/11 \\ 0 & 0 & 79/11 \end{vmatrix} = -11\left(1(1)\left(\frac{79}{11}\right)\right) = -79.$$

Enough to get a triangular matrix
(don't need to go all the way to (R)REF).

Example

$$\begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -2 & 0 \\ 1 & 0 & -1 & 2 \\ 2 & 1 & -2 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & -1 & 2 \\ -1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & -2 & 4 \end{vmatrix}$$

$$= -\begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 0 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & 1 & 0 & 0 \end{vmatrix}$$

$$= \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

Alarm: already know here from 2 identical rows that det = 0

(You could break out of the GJ Elim. rules & do ERO of e.g. $R_3 \to R_3 - R_2$ to get a row of zeros.)

Also happens if one row is a multiple of another row.

(Then you can get a row of zeros with one ERO.)

(*) There are also Elementary Column Operations

→ exactly same as EROs but for columns!

→ have exactly analogous effect on determinant

e.g. swap 2 columns → change sign ($\pm$) of determinant.

(Aside : $\det(A) = \det(A^T)$.)

So above comment also works for columns:

look for columns of zeros, columns that are multiples of other columns. (Then det = 0.)

Can also mix & match all these ideas (if given free choice in finding determinant):

Example

$$\begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 2 & 0 & -2 & 1 \\ 3 & -2 & 3 & -1 \end{vmatrix} \underset{\text{ECO}}{=} \overset{\text{Almost (lower) triangular}}{\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & 2 & 6 & -1 \end{vmatrix}} \leftarrow \text{ERO}$$

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 5 & 2 & 6 & 0 \\ 3 & 2 & 6 & -1 \end{vmatrix}$$

$$= 1(-1)(6)(-1) = 6.$$

2.3 Properties of Determinants

$A \xrightarrow[\substack{\text{(elem.}\\ \text{mat. } E)}]{\text{ERO}} B = EA \Rightarrow \det(B) = \det(E)\det(A)$

& $\det(E) \neq 0$ if E elem.

Recall: A invertible $\iff A = E_1 E_2 \cdot \ldots \cdot E_n$ (some product of elem. matrices.)

elem.

So $\det(A) = \det(E_1) \cdot \ldots \cdot \det(E_n)$ (math. induction!)

$= \det(E_1)\det(E_2 \cdots E_n) = \ldots$ by above

So A invertible $\Rightarrow \det(A) \neq 0$

Recall: A not invertible, then RREF of A has a row of zeros so its det. is 0 hence $\det(A) = 0$. as EROs can't change that.

Also: for any $n \times n$ matrices A, B

$$\det(AB) = \det(A)\det(B)$$

(similar reasoning). See textbook, Theorem 2.3.4.

In particular $\det(I) = \det(A)\det(A^{-1})$ (when A invertible)

$\hookrightarrow \det(A^{-1}) = \dfrac{1}{\det(A)}$

What about A $n \times n$, $k \neq 0$

$$kA = \begin{bmatrix} k \times \text{1st row of } A \\ k \times \text{2nd row of } A \\ \vdots \\ k \times n\text{th row of } A \end{bmatrix}$$

so $\det(kA) = k^{n} \det(A)$.

BAD NEWS: $\det(A+B) \neq \det(A) + \det(B)$ usually

Example $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$ $B = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$

$\det(A) = -1$ $\det(B) = 1$

$A+B = \begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix}$ $\det(A+B) = -3-(-1) = -2$
$\neq -1+1$
$= 0$

But small compensation:

If A, B, C are all $n \times n$ are all equal except for one row (say Row i) and

Row i of C = Row i of A + Row i of B

Then $\det(C) = \det(A) + \det(B)$.

[Can also substitute "column" here for "row".]

<u>Example</u> $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 6 \\ 1 & 3 \end{bmatrix}$ $C = \begin{bmatrix} 1 & 11 \\ 1 & 3 \end{bmatrix}$

$\det(A) = 6 - 5 = 1$ $\det(B) = -3 - 6 = -9$

$\det(C) = 3 - 11 = -8 = 1 - 9$

Here's another Example :

$A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 5 \\ -3 & 2 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -3 \\ -3 & 2 & -2 \end{bmatrix}$ $C = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & 2 \\ -3 & 2 & 0 \end{bmatrix}$

$\det(A) = 2(-10) - (-1)(2 - (-15))$
$= -20 + 17 = -3$
(along first row)

$\det(B) = -(1)(2 - (-2))$
$-(-3)(4-3)$
$= -4 + 3 = -1$
(along 2nd row)

$\det(C) = (-1)(2)$
$- 2(4-3)$
$= -2 - 2 = -4$
$= -3 + (-1)$
(down 3rd column)

Remember, use

$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$ to get your signs correct!