1B03 – LINEAR ALGEBRA 1 (C01) WS19 Lecture 13

Recall Given an $n \times n$ matrix A, the COFACTOR of a_{ij} is $C_{i,j} = (-1)^{i+j} M_{ij}$,

where
$$M_{ij} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & & & & & \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & & & & \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{vmatrix}.$$

Using determinants and cofactors to find inverses!

The adjoint matrix of A (adjugate matrix of A) is

$$\operatorname{adj}(A) = \begin{bmatrix} C_{1,1} & C_{1,2} & \cdots & C_{1,n} \\ C_{2,1} & C_{2,2} & \cdots & C_{2,n} \\ \vdots & & & \\ C_{n,1} & C_{n,2} & \cdots & C_{n,n} \end{bmatrix}^T$$

(the transpose of the matrix of cofactors of A).

Fact If A is invertible, then $A^{-1} = \frac{1}{\det(A)} \operatorname{adj}(A)$.

2×2 case $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ has cofactor matrix

$$\begin{bmatrix} d & -c \\ -b & a \end{bmatrix}$$

$$\begin{bmatrix} + & - \\ - & + \end{bmatrix}$$

& $\det(A) = ad - bc$

So $A^{-1} = \dfrac{1}{ad-bc} \begin{bmatrix} d & -c \\ -b & a \end{bmatrix}^T$

$$= \frac{1}{ad-bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

3×3 case No nice expression.

Example $A = \begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 1 \\ -3 & 2 & 2 \end{bmatrix}$. Find A^{-1}!

Solution Cofactor matrix $= \begin{bmatrix} 0 & -3 & 3 \\ 6 & 12 & -3 \\ -3 & -3 & 3 \end{bmatrix}$

$C_{1,1} = \begin{vmatrix} 1 & 1 \\ 2 & 2 \end{vmatrix} = 2-2=0$

$C_{1,2} = -\begin{vmatrix} 0 & 1 \\ -3 & 2 \end{vmatrix} = -(0-(-3)) = -3$

Check the rest!

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

$C_{2,3} = -\begin{vmatrix} 3 & -1 \\ -3 & 2 \end{vmatrix} = -(6-3) = -3$

So $\operatorname{adj}(A) = \begin{bmatrix} 0 & 6 & -3 \\ -3 & 12 & -3 \\ 3 & -3 & 3 \end{bmatrix}$

Get $\det(A)$ using cofactors:

$$\det(A) = 3(0) + (-1)(-3) + 2(3) = 9.$$

$$A^{-1} = \frac{1}{9}\begin{bmatrix} 0 & 6 & -3 \\ -3 & 12 & -3 \\ 3 & -3 & 3 \end{bmatrix}.$$

Check

$$A^{-1}A = \frac{1}{9}\begin{bmatrix} 0 & 6 & -3 \\ -3 & 12 & -3 \\ 3 & -3 & 3 \end{bmatrix}\begin{bmatrix} 3 & -1 & 2 \\ 0 & 1 & 1 \\ -3 & 2 & 2 \end{bmatrix}$$

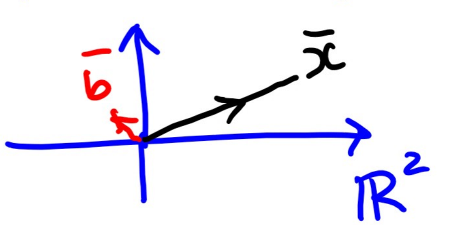
$$= \frac{1}{9}\begin{bmatrix} 9 & 0 & 0 \\ 0 & 9 & 0 \\ 0 & 0 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

5.1 Eigenvalues & Eigenvectors

__Recall__: we write systems of L.E.s now as

$$\underset{m\times n}{A}\;\underset{n\times 1}{\bar{x}} = \underset{m\times 1}{\bar{b}}$$

If $m = n$, this says A "turns" vector $\bar{x}$ into vector $\bar{b}$; A "distorts" $\mathbb{R}^n$



(Some A very special — see Linear Alg. 2)

For now given $n \times n$ A we want to find the vectors $\bar{x} \neq \bar{0}$ which <u>don't</u> get rotated by A (i.e. direction of $\bar{x}$ stays on same line through origin).

<u>Definition</u> If A is $n \times n$, then $\bar{x} \neq \bar{0}$ is an <u>eigenvector</u> of A if $A\bar{x} = \lambda \bar{x}$ for some scalar λ, which is called an <u>eigenvalue</u> of A. (We say "$\bar{x}$ corresponds to λ".)

<u>Examples</u>

- $$\underbrace{\begin{bmatrix} -1 & 2 & 1 \\ 2 & 1 & 0 \\ 1 & 3 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}}_{\bar{x}} = \begin{bmatrix} -3-4+1 \\ 6-2+0 \\ 3-6+1 \end{bmatrix} = \begin{bmatrix} -6 \\ 4 \\ -2 \end{bmatrix} = -2\begin{bmatrix} 3 \\ -2 \\ 1 \end{bmatrix}.$$

- $$\begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 10 \\ 5 \end{bmatrix} = 5\begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

<u>Example</u> Find <u>all</u> eigenvectors & eigenvalues of

$$A = \begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix}.$$

<u>Solution</u> If we write down $A\bar{x} = \lambda \bar{x}$ we get

$$\begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix}$$

i.e. $$\begin{bmatrix} x + 8y \\ 2x + y \end{bmatrix} = \begin{bmatrix} \lambda x \\ \lambda y \end{bmatrix} \quad \ldots \; ?$$

In general, in practice, find <u>eigenvalue(s)</u> λ first.
(Then find corresponding eigenvectors.)

How? Want λ, $\bar{x} \neq \bar{0}$ with $A\bar{x} = \lambda \bar{x}$

$$= \underbrace{\lambda I}_{\begin{bmatrix} \lambda & & 0 \\ & \ddots & \\ 0 & & \lambda \end{bmatrix}} \bar{x}$$

$$\Rightarrow \; A\bar{x} - \lambda I \bar{x} = \bar{0}$$

$$\Rightarrow \; \underbrace{(A - \lambda I)} \bar{x} = \bar{0}$$

We want $\bar{x} \neq \bar{0}$ solving ↑ this system.

So we need $A - \lambda I$ to be NOT invertible.

This happens exactly when <u>$\det(A - \lambda I) = 0$</u>.

This is the <u>characteristic equation</u> for A.

This does not involve $\bar{x}$. So solve this eq. for λ & then go back & find $\bar{x} \neq \bar{0}$ with $A\bar{x} = \lambda \bar{x}$