1B03 – LINEAR ALGEBRA 1 (C01) WS19 Lecture 14

Last time: The Eigenvalue/Eigenvector Problem

GOAL: Given A $n \times n$, find $\bar{x} \neq \bar{0}$ and λ with

$$A\bar{x} = \lambda \bar{x}$$

eigenvalue of A (scalar) — λ; eigenvector of A (vector!) — $\bar{x}$

Find λ first by solving the characteristic equation of A:

$$\boxed{\det(A - \lambda I) = 0}$$

Example Find eigenvalues & eigenvectors of $A = \begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix}$

Solution Set

$$\begin{aligned}
0 &= \det\left(\begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix} - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \right) \\
&= \det\left(\begin{bmatrix} 1-\lambda & 8 \\ 2 & 1-\lambda \end{bmatrix} \right) \\
&= (1-\lambda)^2 - 16 \\
&= 1 - 2\lambda + \lambda^2 - 16 \\
&= \lambda^2 - 2\lambda - 15 \\
&= (\lambda - 5)(\lambda + 3)
\end{aligned}$$

So eigenvalues of A are $\lambda = 5$ and -3

Notice $\underbrace{\det(A-\lambda I)}$ is always a polynomial.

Called characteristic polynomial

To find eigenvectors, now we have identified the eigenvalues, we have solve $\underbrace{A\bar{x} = \lambda\bar{x}}$ for each eigenvalue λ that we found

OR

$(A-\lambda I)\bar{x} = \bar{0}$

For example: $\lambda = 5$ $\qquad A = \begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix}$

Solve $(A - 5I)\bar{x} = \bar{0}$

$$\begin{bmatrix} \cancel{1-5}^{-4} & 8 \\ 2 & \cancel{1-5}_{-4} \end{bmatrix} \bar{x} = \bar{0}$$

Remember: the augmented matrix of a system $A\bar{x} = \bar{b}$ is $[A \mid b]$.

$a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n = b_1$
$a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n = b_2$
$\vdots$
$a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n = b_n$

Aug. matrix

$$\left[\begin{array}{cc|c} -4 & 8 & 0 \\ 2 & -4 & 0 \end{array}\right] \xrightarrow[\text{RREF}]{\text{Reduce to}} \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 2 & -4 & 0 \end{array}\right]$$

$$\to \left[\begin{array}{cc|c} 1 & -2 & 0 \\ 0 & 0 & 0 \end{array}\right]$$

$y = t$

: Solutions

$x - 2y = 0$

$x - 2t = 0$

$x = 2t$

i.e. $\begin{pmatrix} 2t \\ t \end{pmatrix}$

So eigenvectors for $\lambda = 5$ are $\begin{pmatrix} 2t \\ t \end{pmatrix}$ for $\underbrace{t \neq 0}$.

If we put $t=0$, we get a solution to the system, $\bar{x}=\bar{0}$, but $\bar{x}=\bar{0}$ NOT an eigenvector.

$\underline{\lambda = -3}$: solve $(A-(-3)I)\bar{x} = \bar{0}$

i.e. $(A+3I)\bar{x} = \bar{0}$

i.e. $\left(\begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix} + \begin{bmatrix} 3 & 0 \\ 0 & 3 \end{bmatrix}\right)\bar{x} = \bar{0}$

i.e. $\begin{bmatrix} 4 & 8 \\ 2 & 4 \end{bmatrix}\bar{x} = \bar{0}$.

Aug. matrix

$$\left[\begin{array}{cc|c} 4 & 8 & 0 \\ 2 & 4 & 0 \end{array}\right] \xrightarrow[\text{to RREF}]{\text{Reduce}} \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 2 & 4 & 0 \end{array}\right]$$

$$\longrightarrow \left[\begin{array}{cc|c} 1 & 2 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow x + 2y = 0$$

$y = t$ $\qquad x = -2t$

Solutions : $\begin{pmatrix} -2t \\ t \end{pmatrix}$

Eigenvectors for $\lambda = -3$ are $\begin{pmatrix} -2t \\ t \end{pmatrix}$ for $t \neq 0$.

__Notice__ : Always get infinitely many eigenvectors:

In general if $\bar{x} \neq 0$, λ satisfy : $A\bar{x} = \lambda\bar{x}$,

then $A(k\bar{x}) = k(A\bar{x}) = k(\lambda\bar{x}) = \lambda(k\bar{x})$

$k \neq 0$ scalar

So if $\bar{x}$ is an eigenvector corresp. to λ, so is $k\bar{x}$ $(k \neq 0)$

In our example eigenvectors of $\lambda = 5$ were $\begin{pmatrix} 2t \\ t \end{pmatrix}$ $(t \neq 0)$

& for $\lambda = -3$ were $\begin{pmatrix} -2t \\ t \end{pmatrix}$ $(t \neq 0)$.

For a given eigenvalue λ, the collection of <u>all</u> eigenvectors corresp. to λ together with $\bar{0}$ is called the <u>eigenspace of λ</u>.

Eigenspaces have a <u>basis</u> : a list (maybe only one) of eigen vectors in the eigenspace that "<u>describe</u>" all the eigenvectors in the eigenspace.

More formally: Say basis is $\bar{x}_1, \bar{x}_2, \ldots, \bar{x}_k$; ~~and~~ all eigenvectors have form

$$\underbrace{t_1 \bar{x}_1 + t_2 \bar{x}_2 + \ldots + t_k \bar{x}_k}_{\text{"linear combination of } \bar{x}_1, \ldots, \bar{x}_k\text{"}}$$

t_i : scalars

For the example, eigenspace of $\lambda = 5$ is

$\left\{ \begin{pmatrix} 2t \\ t \end{pmatrix} \middle| t \in \mathbb{R} \right\}$ which has as a basis e.g. $\left\{ \begin{pmatrix} 2 \\ 1 \end{pmatrix} \right\}$

$\begin{pmatrix} 2t \\ t \end{pmatrix} = t \begin{pmatrix} 2 \\ 1 \end{pmatrix}$

The eigenspace of $\lambda = -3$ is $\left\{ \begin{pmatrix} -2t \\ t \end{pmatrix} \middle| t \in \mathbb{R} \right\}$

which has as a basis e.g. $\left\{ \begin{pmatrix} -2 \\ 1 \end{pmatrix} \right\}$.

$\begin{pmatrix} -2t \\ t \end{pmatrix} = t \begin{pmatrix} -2 \\ 1 \end{pmatrix}$.

__Question__ Can eigenvalues λ be 0?

This means $\boxed{A\bar{x} =} 0\bar{x} = \boxed{\bar{0}}$ for some $\bar{x} \neq \bar{0}$.

__Answer__ Yes, λ can be 0 & in fact it tells us about A.

It tells us A NOT invertible.

(__Remember__: A invertible $\iff$ the only solution to $A\bar{x}=\bar{0}$ is $\bar{x}=\bar{0}$.)

So for A $n \times n$, A invertible $\iff$

$\lambda = 0$ is NOT eigenvalue of A

$\iff$ $\lambda = 0$ is NOT a root of the characteristic equation

$$\det(A - \lambda I) = 0.$$

__Special cases__ Suppose $A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ 0 & a_{22} & \cdots & a_{2n} \\ \vdots & & \ddots & \\ 0 & 0 & \cdots 0 & a_{nn} \end{bmatrix}$ ← upper $\triangle$

What are the eigenvalues of A?

Solve $\det(A - \lambda I) = 0$.

$$\det \begin{bmatrix} a_{11}-\lambda & a_{12} & \cdots & a_{1n} \\ 0 & a_{22}-\lambda & \cdots & a_{2n} \\ \vdots & & & \\ 0 & 0 & \cdots\ 0 & a_{nn}-\lambda \end{bmatrix}$$

upper Δ

$$= (a_{11}-\lambda)(a_{22}-\lambda)\cdots(a_{nn}-\lambda)$$

So eigenvalues are $a_{11}, a_{22}, \dots, a_{nn}$.

In general, the eigenvalues of a triangular (or diagonal) matrix are the entries of the leading diagonal.

Example Find eigenvalues of $\begin{bmatrix} 3 & 6 & 9 \\ 0 & 2 & -5 \\ 0 & 0 & -1 \end{bmatrix}$.

$$\lambda = 3, 2, -1.$$

Example $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$. Find eigenvalues & eigenvectors.

Solution

$$0 = \det(A-\lambda I) = \begin{vmatrix} 1-\lambda & 1 \\ 0 & 1-\lambda \end{vmatrix}$$

$$= (1-\lambda)^2$$

So (only) 1 eigenvalue is $\lambda = 1$

Eigenvectors: solve $(A - I)\bar{x} = \bar{0}$

$$\left[\begin{array}{cc|c} 0 & 1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow y = 0$$

$x = t$ Solutions: $\begin{pmatrix} t \\ 0 \end{pmatrix}$

so eigenvectors are given by $\left\{ \begin{pmatrix} 1 \\ 0 \end{pmatrix} \right\}$ basis for eigenspace $\therefore$ all eigenvectors are of form $t\begin{pmatrix} 1 \\ 0 \end{pmatrix}$. $\left(= \begin{pmatrix} t \\ 0 \end{pmatrix}\right)$

(# eigenvalues) & # eigenvectors is important but not obvious!

For example $A = \begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix}$, # eigenvalues = 2

2x2

for each eigenspace, # basis vectors = 1

For example $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$, # eigenvalues = 1 &

2x2

for the only eigenspace, # basis vectors = 1.

What the # eigenvalues & # eigenvectors of a matrix A depends on is not at all clear... but when the "right" numbers appear, magic happens!