1B03 – LINEAR ALGEBRA 1 (C01) WS19 Lecture 15

Last Time More on the Eigenvalue/Eigenvector Problem

To find $\bar{x} \neq \bar{0}$ and λ with $A\bar{x} = \lambda\bar{x}$:

(1) First find all possible λ by solving $\det(A - \lambda I) = 0$ for λ

(2) For each of the λ-values found in (1), solve $(A - \lambda I)\bar{x} = \bar{0}$ for $\bar{x}$.

Eigenspace of λ : All eigenvectors $\bar{x} \neq \bar{0}$ corresponding to λ + the zero vector $\bar{0}$

Basis for Eigenspace: Some list of eigenspace vectors $\bar{x}_1, \ldots, \bar{x}_k$ that gets you all eigenspace vectors by linear combination: $t_1\bar{x}_1 + \ldots + t_k\bar{x}_k$.

Last time:

Example 1 $\quad A : \begin{bmatrix} 1 & 8 \\ 2 & 1 \end{bmatrix} \quad$ 2 eigenvalues $\lambda = -3, 5$

For $\lambda = -3$ eigenvectors $\begin{pmatrix} -2t \\ t \end{pmatrix} = t\begin{pmatrix} -2 \\ 1 \end{pmatrix}$ so basis e.g. $\left\{\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right\}$

For $\lambda = 5$ eigenvectors $\begin{pmatrix} 2t \\ t \end{pmatrix} = t\begin{pmatrix} 2 \\ 1 \end{pmatrix}$. so basis e.g. $\left\{\begin{pmatrix} 2 \\ 1 \end{pmatrix}\right\}$

Example 2 $\quad A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad$ 1 eigenvalue $\lambda = 1$

with eigenvectors $\begin{pmatrix} t \\ 0 \end{pmatrix} = t\begin{pmatrix} 1 \\ 0 \end{pmatrix}$

So e.g. basis $\left\{\begin{pmatrix} 1 \\ 0 \end{pmatrix}\right\}$.

Example 3 $\quad A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 4 & 3 \end{bmatrix}.$

To find eigenvalues solve $0 = \begin{vmatrix} 5-\lambda & 0 & 0 \\ 0 & 1-\lambda & 2 \\ 0 & 4 & 3-\lambda \end{vmatrix}$

$$= (5-\lambda) \begin{vmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{vmatrix} = (5-\lambda)\big((1-\lambda)(3-\lambda) - 8\big)$$

$$= (5-\lambda)(3 - 4\lambda + \lambda^2 - 8)$$

$$= (5-\lambda)(\lambda^2 - 4\lambda - 5)$$

$$= (5-\lambda)(\lambda+1)(\lambda-5)$$

So eigenvalues are $\lambda = -1, 5$ (5 is a "repeated eigenvalue")

$\lambda = -1$ Solve $A\bar{x} = -\bar{x}$ i.e. $(A+I)\bar{x} = \bar{0}$

$$\left[\begin{array}{ccc|c} 6 & 0 & 0 & 0 \\ 0 & 2 & 2 & 0 \\ 0 & 4 & 4 & 0 \end{array}\right] \xrightarrow{\text{Check!}} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

$z = t$

$x = 0$

$y + z = 0$ i.e. $y = -z = -t$

Eigenvectors are $\begin{pmatrix} 0 \\ -t \\ t \end{pmatrix}$ $(t \neq 0)$ $= t\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}$ so choose basis $\left\{\begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix}\right\}$

$\underline{\lambda = 5}$ Solve $A\bar{x} = 5\bar{x}$ i.e. $\underline{(A-5I)\bar{x} = \bar{0}}$

$$\left[\begin{array}{ccc|c} 0 & 0 & 0 & 0 \\ 0 & -4 & 2 & 0 \\ 0 & 4 & -2 & 0 \end{array}\right] \xrightarrow{\text{Check!}} \left[\begin{array}{ccc|c} 0 & 1 & -\frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right] \begin{array}{l} \rightarrow y - \frac{1}{2}z = 0 \\ \Rightarrow y = \frac{z}{2} \\ = \frac{s}{2} \end{array}$$

$x = t \qquad z = s$

So eigenvectors: $\begin{pmatrix} t \\ s/2 \\ s \end{pmatrix}$ (t & s not both 0)

$$= t\underline{\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}} + s\underline{\begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix}}$$

So basis is e.g. $\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix} \right\}$.

Multiplicities of eigenvalues λ

Algebraic Multiplicity of λ:

times λ appears as a root in the characteristic polynomial

(In previous example $AM(-1) = 1$ & $AM(5) = 2$.)

Geometric Multiplicity of λ:

basis vectors in a basis for eigenspace for λ

= # free variables in solutions to $(A - \lambda I)\bar{x} = \bar{0}$

$= \#$ non-pivot columns in RREF of $A - \lambda I$.

Fact

$$1 \leq GM(\lambda) \leq AM(\lambda).$$

[Can't get more basis vectors for λ than # of times λ appears as a root.]

Cool things happen when

$$GM(\lambda) = AM(\lambda).$$

Say A is $n \times n$. We're interested in when total # of basis vectors (across all λ) $= n$.

$= \text{sum of } GM(\lambda)$

This definitely happens when all roots of the characteristic polynomial are distinct.

i.e. no repeated roots so for every λ

$$\underset{\substack{\vee\!\shortmid \\ 1}}{GM(\lambda)} \leq AM(\lambda) = 1 \qquad \text{so} \quad AM(\lambda) = GM(\lambda) = 1$$

The characteristic polynomial of an $n \times n$ matrix has degree n so we get n roots hence in this case n basis vectors.

5.2 Diagonalization

Let A be $n \times n$ with eigenvalues $\lambda_1, \dots, \lambda_k$.

Suppose we have n (different) basis vectors (i.e. $n = GM(\lambda_1) + GM(\lambda_2) + \dots + GM(\lambda_k)$).

Then do the following:

Set $P = \begin{bmatrix} | & | & & | \\ P_1 & P_2 & \cdots & P_n \\ | & | & & | \end{bmatrix}$ where columns $P_1, \dots, P_n$ are the n basis vectors (in whatever order)

$n \times n$

Then <u>Fact</u> $D = P^{-1} A P$ is diagonal and in fact the diagonal entries d_{ii} = eigenvalue to which P_i corresponds

<u>Example</u> Above (Ex. 3) $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 4 & 3 \end{bmatrix} \leftarrow 3 \times 3$

Eigenvalues λ:	-1	5
# basis vectors:	1	2
basis vectors:	$\left\{ \begin{pmatrix} 0 \\ -1 \\ 1 \end{pmatrix} \right\}$	$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix} \right\}$

So now set $P = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 1/2 \\ 1 & 0 & 1 \end{bmatrix}$.

Then $P^{-1}AP = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 5 \end{bmatrix}$ ← Check!

(If we had made $Q = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 1/2 \\ 0 & 1 & 1 \end{bmatrix}$, then

same columns as P but in a different order

$Q^{-1}AQ = \begin{bmatrix} 5 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 5 \end{bmatrix}$.) ← Check!

So the order in which the eigenvalues appear on the diagonal changes to match the order of the eigenvectors as columns of Q.