1B03 – LINEAR ALGEBRA 1 (C01) WS19 Lecture 16

Last Time . . . we saw that if A is $n \times n$ and

A has n basis vectors across all of its eigenvalues,

[i.e. for each eigenvalue λ of A, $AM(\lambda) = GM(\lambda)$

$AM(\lambda)$: # times λ is a root of $\det(A-\lambda I)$ — $GM(\lambda)$: # vectors in basis for eigenspace of λ]

then $\boxed{D = P^{-1}AP}$ $\leftarrow P = \begin{bmatrix} | & & | \\ p_1 & \cdots & p_n \\ | & & | \end{bmatrix}$ with $Ap_i = \lambda_i p_i$

$\parallel$

diagonal: $\begin{bmatrix} \lambda_1 & & 0 \\ & \ddots & \\ 0 & & \lambda_n \end{bmatrix}$

Definition For A, B $n \times n$, we say B is similar to A if there is some invertible matrix P with $B = P^{-1}AP$.

Notice: If B similar to A, so there's P invertible with $B = P^{-1}AP$

$$PB = AP$$

$$PBP^{-1} = A \qquad \text{so if we set } Q = P^{-1}$$

this is $Q^{-1}BQ = A$.

So A is also similar to B.

Fact Suppose A & B are similar. Then:

- $\det(A) = \det(B)$
- A invertible $\Leftrightarrow$ B invertible
- $\operatorname{tr}(A) = \operatorname{tr}(B)$
- A & B have the same characteristic polynomial
  $\det(A - \lambda I) = \det(B - \lambda I)$
- A & B have same eigenvalues
- If λ is an eigenvalue of A (& so also of B) then # basis vectors in eigenspace of A corresponding to λ = # basis vectors in eigenspace of B corresponding to λ.

Definition If A is similar to a diagonal matrix D, then we say that A is **diagonalizable**

i.e. there's an invertible P with
$$D = P^{-1}AP$$
– we say that P **diagonalizes** A

So Fact from last time can be reframed:

Fact If A is $n \times n$ & has n different basis vectors for its eigenspaces, then A is

<u>diagonalizable</u> : $D = P^{-1} A P$

diagonal entries eigenvalues of A

columns basis vectors corresponding to eigenvalues

It turns out the reverse is true:

Say A, P $n \times n$ with $\boxed{P^{-1}AP = D}$ and D diagonal.

Then $AP = PD$

$$A \begin{bmatrix} | & & | \\ P_1 & \cdots & P_n \\ | & & | \end{bmatrix} \qquad \begin{bmatrix} | & & | \\ P_1 & \cdots & P_n \\ | & & | \end{bmatrix} \begin{bmatrix} d_{11} & & 0 \\ & \ddots & \\ 0 & & d_{nn} \end{bmatrix}$$

So for each column P_i of P we have $AP_i = d_{ii} P_i$

i.e. P_i is an eigenvector for A corresponding to $\lambda = d_{ii}$, for each i.

(<u>Note</u>: P invertible, so all columns of P different "enough".)

<u>Fact</u> A $n \times n$ diagonalizable $\iff$ A has n basis vectors across all its eigenspaces

Remember: In particular, if characteristic poly. of A $(\det(A-\lambda I))$ has n distinct roots, then A is diagonalizable.

Strategy for diagonalizing a matrix A (if you can)

A is $n \times n$

(1) Find all eigenvalues of A

(2) Find basis vectors for each eigenspace

Q: Are there n basis vectors in step (2)?

If NO: A not diagonalizable.

If YES:

(3) Write $P = \begin{bmatrix} | & & | \\ P_1 & \cdots & P_n \\ | & & | \end{bmatrix}$ with basis vectors $P_1, \ldots, P_n$

(4) Then $P^{-1}AP = D = \begin{bmatrix} d_{11} & & 0 \\ & \ddots & \\ 0 & & d_{nn} \end{bmatrix}$ where d_{ii} is the eigenvalue to which P_i corresponds.

Special case: If A is $n \times n$ triangular & has distinct diagonal entries ($\Rightarrow$ distinct eigenvalues) then A is diagonalizable.

If there are repeated diagonal entries for triangular matrix A, maybe diagonalizable, maybe not (need to check).

[In general, just to check diagonalizability, start by checking that the repeating eigenvalues have enough basis vectors
times eigenvalue repeats]
i.e. $AM(\lambda) = GM(\lambda)$.

UP TO HERE FOR TEST 1

The power of diagonalizability

A diagonalizable : D diagonal
P invertible

with $D = P^{-1}AP$.

$PDP^{-1} = A$

Want to compute A^k.

This is $A^k = \overbrace{(PD\cancel{P^{-1}})(\cancel{P}D\cancel{P^{-1}})(\cdot \;\ldots\; \cancel{\cdot})(\cancel{P}DP^{-1})}^{k \text{ times}}$

$$= PD^kP^{-1}$$

$$D^k = \begin{bmatrix} d_{11} & & 0 \\ & \ddots & \\ 0 & & d_{nn} \end{bmatrix}^k = \begin{bmatrix} d_{11}^k & & 0 \\ & \ddots & \\ 0 & & d_{nn}^k \end{bmatrix}$$

So PD^kP^{-1} is in general <u>much</u> easier to compute than A^k. But first need to find P, D, P^{-1}.