1B03 – LINEAR ALGEBRA 1 (C01) WS19 Lecture 17

Yesterday

DIAGONALIZATION

A diagonalizable : $P^{-1}AP = D$ ← D diagonal, P invertible

(We might say "A is similar to a diagonal matrix".)

P : columns are basis vectors for eigenspaces of A

D : diagonal entries are eigenvalues of A

POWERS of diagonalizable matrices:

$$A = PDP^{-1} \Rightarrow A^k = PD^kP^{-1}$$

Example $A = \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}$. Compute a general formula for A^k / compute A^{15}.

Eigenvalues:

$$0 = \begin{vmatrix} 3-\lambda & -1 \\ 2 & -\lambda \end{vmatrix} = (3-\lambda)(-\lambda) - (-2) = \lambda^2 - 3\lambda + 2$$
$$= (\lambda-2)(\lambda-1)$$

$\lambda = 1, 2.$ (So $D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$ or $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.)

For $\lambda = 1$: solve $(A - I)\bar{x} = \bar{0}$

$$\left[\begin{array}{cc|c} 2 & -1 & 0 \\ 2 & -1 & 0 \end{array}\right] \to \left[\begin{array}{cc|c} 1 & -1/2 & 0 \\ 0 & 0 & 0 \end{array}\right] \to \text{eigenvectors } t\begin{pmatrix} 1/2 \\ 1 \end{pmatrix}$$

for $\lambda = 2$: solve $(A - 2I)\bar{x} = \bar{0}$

$$\rightsquigarrow \left[\begin{array}{cc|c} 1 & -1 & 0 \\ 0 & 0 & 0 \end{array}\right] \rightarrow \text{eigenvector } t\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

So $\boxed{P = \begin{bmatrix} 1/2 & 1 \\ 1 & 1 \end{bmatrix} \text{ (corresponds to } D = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}}$ or $P = \begin{bmatrix} 1 & 1/2 \\ 1 & 1 \end{bmatrix}$ corresponds to $D = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$.)

Then $A = PDP^{-1} \longrightarrow$ find $P^{-1} = \begin{bmatrix} -2 & 2 \\ 2 & -1 \end{bmatrix}$.

So now we have

$$A^k = PD^kP^{-1} = \begin{bmatrix} 1/2 & 1 \\ 1 & 1 \end{bmatrix}\begin{bmatrix} 1^k & 0 \\ 0 & 2^k \end{bmatrix}\begin{bmatrix} -2 & 2 \\ 2 & -1 \end{bmatrix}$$

$$\vdots$$

$$= \begin{bmatrix} 2^{k+1} - 1 & 1 - 2^k \\ 2^{k+1} - 2 & 2 - 2^k \end{bmatrix}$$

So if $k = 15$, then $A^{15} = \begin{bmatrix} 2^{16} - 1 & 1 - 2^{15} \\ 2^{16} - 2 & 2 - 2^{15} \end{bmatrix}$

$$= \begin{bmatrix} 65{,}535 & -32{,}767 \\ 65{,}534 & -32{,}766 \end{bmatrix}.$$

5.4 Differential Equations (DEs)

→ We're looking at DEs of form

$$\boxed{y' = ky}$$

(k constant)

$y = f(t)$

$$\Rightarrow \int \frac{y'}{y} = \int k \Rightarrow \ln|y| = kt + C$$

$$\Rightarrow |y| = e^{kt+C}$$

$$\Rightarrow y = \pm e^{kt+C}$$

$$\Rightarrow \boxed{y = y_0 e^{kt}} \qquad (y_0 = \pm e^C)$$

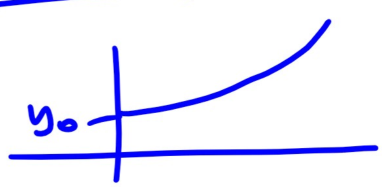
↑ General solution

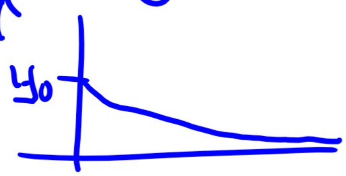
With extra info.: the value of y at 0 i.e. $f(0)$

$$y(0) = y_0 e^{k \cdot 0} = y_0 \cdot 1$$

Motivation

Exponential growth ($k > 0$) or decay ($k < 0$)





Appears:
- Wealth (compound interest)
- Spread of infection
- Populations

- Radioactive Decay
- Metabolism of compounds

Systems of (these) DEs

e.g. • Interaction of populations with rate of change dependent on all pop. sizes

Example ① Zombie apocalypse.

Human pop. $h(t)$, Zombie pop. $z(t)$

could be modelled by
$$h' = 3h - z$$
$$z' = 2h.$$

② We can simplify things like: $y''' + 3y'' - 2y' + 6y = 0$

(weighted sum of higher order derivatives)

Rename: $y_1 = y$

$\boxed{y_2 = y_1'}$ $(= y')$

$\boxed{y_3 = y_2'}$ $(= y'')$

System of (these) DEs.

$y_3' + 3y_3 - 2y_2 + 6y_1 = 0$

i.e.

$$\boxed{y_3' = -6y_1 + 2y_2 - 3y_3}$$

How to solve systems of (these) DEs?

<u>1st big idea</u> (Lin. Alg.) We write them:

$$\begin{bmatrix} y_1' \\ \vdots \\ y_n' \end{bmatrix} = \bar{y}' = A\bar{y} = A \begin{bmatrix} y_1 \\ \vdots \\ y_n \end{bmatrix}$$

$n \times n$ matrix of coefficients

In examples above:

① (zombies) $\bar{y} = \begin{bmatrix} h \\ z \end{bmatrix}$; $\bar{y}' = \begin{bmatrix} h' \\ z' \end{bmatrix} = \begin{bmatrix} 3h - z \\ 2h \end{bmatrix}$

$$= \begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix} \bar{y}.$$

② $\bar{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$ $\quad \bar{y}' = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -6 & 2 & -3 \end{bmatrix} \bar{y}.$

Suppose we have $\bar{y}' = A\bar{y}$.

<u>Notice</u> If A were diagonal, then this is just

$$\left.\begin{aligned} y_1' &= a_{11} y_1 \\ y_2' &= a_{22} y_2 \\ &\vdots \\ y_n' &= a_{nn} y_n \end{aligned}\right\}$$

<u>uncoupled equations</u>.

- solve each separately

→ Solution : $y_1 = y_{1,0} e^{a_{11} t}$ $\quad y_{i,0} = y_i(0)$

$\vdots \qquad \vdots$

$y_n = y_{n,0} e^{a_{nn} t}$

<u>2nd big idea</u> If A diagonalizable, we can do something similar.

$$A = PDP^{-1} \quad (D \text{ diagonal}).$$

$$\bar{y}' = A\bar{y} = (PDP^{-1})\bar{y} \Rightarrow P^{-1}\bar{y}' = D(P^{-1}\bar{y})$$

Make a change of variables : let $\bar{u} = P^{-1}\bar{y}$

Then $\bar{u}' = D\bar{u}$.

We know solution : $u_i = \underbrace{u_{i,0}}\, e^{d_{ii} t}$

$u_{i,0} = u_i(0)$

Initial value of u_i

↳ get these from $\bar{u}(0) = (P^{-1}\bar{y})(0)$

$$\begin{bmatrix} u_{1,0} \\ \vdots \\ u_{n,0} \end{bmatrix} = P^{-1} \begin{bmatrix} y_{1,0} \\ \vdots \\ y_{n,0} \end{bmatrix}$$

Now our solution is $\boxed{\bar{y} = P\bar{u}.}$

Recall: Column P_i of P is eigenvector corresponding to eigenvalue d_{ii}

Can rewrite our solution: $\bar{y} = \begin{bmatrix} | & & | \\ p_1 & \cdots & p_n \\ | & & | \end{bmatrix} \bar{u}$

$$= \begin{bmatrix} | & & | \\ p_1 & \cdots & p_n \\ | & & | \end{bmatrix} \begin{bmatrix} u_{1,0}\, e^{d_{11}t} \\ \vdots \\ u_{n,0}\, e^{d_{nn}t} \end{bmatrix}$$

$$\Rightarrow \boxed{\bar{y} = u_{1,0} P_1 e^{d_{11}t} + \dots + u_{n,0} P_n e^{d_{nn}t}}$$

eigenvector eigenvalue.

Example ① $\bar{y}' = \underbrace{\begin{bmatrix} 3 & -1 \\ 2 & 0 \end{bmatrix}}_{A} \bar{y}$ with initially 1000 humans & 75 zombies

Solution

$$A = PDP^{-1}$$

$$= \begin{bmatrix} 1/2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -2 & 2 \\ 2 & -1 \end{bmatrix}$$

P_1 P_2 d_{11} d_{22} P^{-1} ← From above.

$$\bar{y}_0 = \begin{bmatrix} 1000 \\ 75 \end{bmatrix}$$

Find $\bar{u}_0 = P^{-1}\bar{y}_0 = \begin{bmatrix} -2 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} 1000 \\ 75 \end{bmatrix} = \begin{bmatrix} -1850 \\ 1925 \end{bmatrix}$

So $\bar{y} = u_{1,0} P_1 e^{d_{11}t} + u_{2,0} P_2 e^{d_{22}t}$

$$= -1850 \begin{bmatrix} 1/2 \\ 1 \end{bmatrix} e^{t} + 1925 \begin{bmatrix} 1 \\ 1 \end{bmatrix} e^{2t}.$$