1B03 – LINEAR ALGEBRA 1 (C01) WS19 Lecture 18

Today **Complex Numbers**

See Chapter 10 of 9th Edition of Anton & Rorres textbook.
($\longrightarrow$ find on Course Website.)

Example What are the eigenvalues of $A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$?

Solve $0 = \det(A - \lambda I) = \begin{vmatrix} 2-\lambda & -5 \\ 1 & -2-\lambda \end{vmatrix} = (2-\lambda)(-2-\lambda)+5$

$= -4 \cancel{+2\lambda} \cancel{-2\lambda} + \lambda^2 + 5$

$= \lambda^2 + 1$

We want λ with $\lambda^2 = -1$

Isn't $x^2 \geq 0$ for all x?? Yes, if x is real.

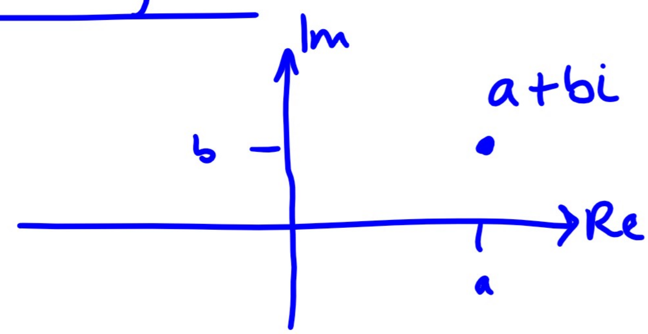
Need new numbers!

Definitions The imaginary unit i is the number satisfying $\boxed{i^2 = -1.}$

A **complex** (AKA **imaginary**) **number** is a number of the form $z = a + bi$ where a, b real #s

a called real part of z; b called imaginary part of z.

Visualize complex #s as vectors (a,b) in **Complex Plane** (AKA **Argand Diagram**) with **real** and **imaginary axes**:

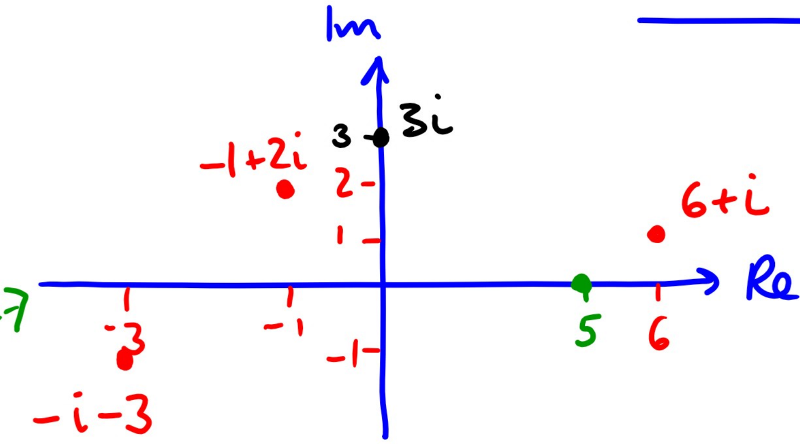


Examples

$6 + i$

$-1 + 2i$

$-i - 3$



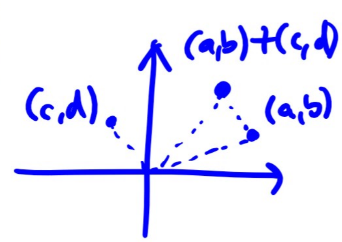
$3i$ – called **purely imaginary**

5 – called **real**!

Operations on Complex Numbers

- **Addition / Subtraction** $(a + bi) + (c + di)$

Vectors: $(a,b) + (c,d)$
$= (a+c,\ b+d)$



$= (a+c) + (b+d)i$

$$= a + bi + c + di$$
$$= a + c + bi + di$$
$$= (a+c) + (b+d)i$$

<u>Example</u> $(\underline{1} - \underline{i}) + (\underline{6} + \underline{3i}) = 7 + 2i.$

- <u>Multiplication</u> $(a+bi)(c+di) = ac + bci + adi \boxed{+bdi^2}$ $\quad \boxed{+bdi^2} = -bd$

(No good vector analogue (unless $b=0$) — at least not the way we write vectors so far — see next time!)

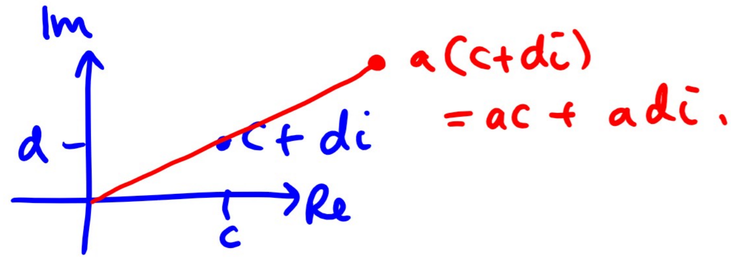
$$= (ac - bd) + (bc + ad)i.$$

<u>Example</u> $(1-i)(6+3i) = 6 - 6i + 3i \boxed{-3i^2}$ $\quad \boxed{-3i^2} = +3$

$$= 9 - 3i.$$

If $b = 0$ so we have $a(\underbrace{c+di}) = ac + adi$

"Scalar multiplication of a complex # $(c+di)$ by a real # (a)"



<u>Example</u> $\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right)^2 = \left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right)\left(\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i\right)$

$$= \ldots$$

$$= \left(\frac{1}{\sqrt{2}}\right)(1+i)\left(\frac{1}{\sqrt{2}}\right)(1+i) = \frac{1}{2}(1+i)(1+i) = \frac{1}{2}(\cancel{1}+i+i+i^2)$$

$$i^2 = \cancel{-1}$$

$$= \frac{1}{2}(2i) = i.$$

Does this mean $\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$ is a square root of i?

Yes! (More on this next time!)

↳ Actually Lecture 20.

Now that we can add/subtract/multiply complex #s, we can work with <u>vectors & matrices whose entries are complex #s</u> — matrix operations carry over.

<u>Example</u>

$$(1+3i)\begin{bmatrix} 1 & -i \\ -2+i & 0 \end{bmatrix} = \begin{bmatrix} 1+3i & (1+3i)(-i) \\ (1+3i)(-2+i) & 0 \end{bmatrix}$$

Scalar multiplication where the scalar is a complex # & the matrix has complex # entries.

$$= \begin{bmatrix} 1+3i & -3i^2 - i \\ -2-6i+i+3i^2 & 0 \end{bmatrix} = \begin{bmatrix} 1+3i & 3-i \\ -5-5i & 0 \end{bmatrix}.$$

Recall: Matrix $A = \begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix}$ had char. equation

$\lambda^2 + 1 = 0$ so eigenvalues $\lambda = \pm i$.

For example: $\begin{bmatrix} 2 & -5 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} 2+i \\ 1 \end{bmatrix} = \begin{bmatrix} 2(2+i)-5 \\ 2+i-2 \end{bmatrix}$

$$= \begin{bmatrix} -1+2i \\ i \end{bmatrix} = i \begin{bmatrix} 2+i \\ 1 \end{bmatrix}$$

i.e. $\begin{bmatrix} 2+i \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = i$.

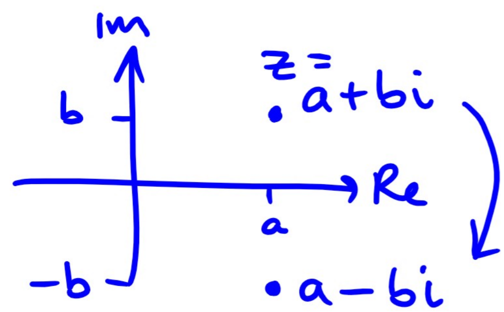
- **Division** (of complex #s) Given $w = a+bi$ & $z = c+di$

What is $\dfrac{w}{z} = \dfrac{a+bi}{c+di}$? $= x + yi$

It's some $x+yi$, where $a+bi = (x+yi)(c+di)$.

To work out the answer we need:

Definitions The complex conjugate of a complex # $z = a+bi$ is $\bar{z} = a - bi$.



Mirror in Real axis.

Examples

$z = 6+5i$; $\bar{z} = 6-5i$

$z = -3-2i$; $\bar{z} = -3+2i$

$z = 4i-1$; $\bar{z} = -4i-1$

$z = 3$; $\bar{z} = 3$; $z = -i$; $\bar{z} = i$

<u>Notice</u> $z + \bar{z} = (a+bi) + (a-bi) = 2a$ ← always <u>real</u>.

$z - \bar{z} = (a+bi) - (a-bi) = 2bi$ ← always <u>pure imaginary</u>.

$$z\bar{z} = (a+bi)(a-bi) = a^2 + abi - abi - b^2 i^2$$
$$= a^2 + b^2.$$

The <u>modulus</u> / <u>absolute value</u> of $z = a+bi$ is

$$|z| = \sqrt{z\bar{z}} \quad \left(= \sqrt{a^2+b^2} \right)$$

(So we showed $z\bar{z} = |z|^2$ $(= a^2+b^2)$.)

Notice: if $b=0$ (so $z = a$, real), then

$|z| = \sqrt{a^2} = |a|$ (so this agrees with our definition for real #s!)