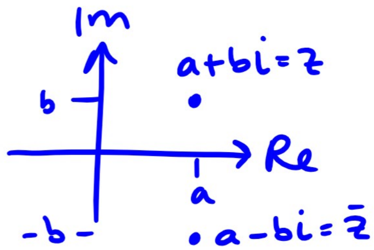
# 1B03 - LINEAR ALGEBRA 1 (C01) WS19 Lecture 19

## Last Time

**Complex Numbers**




$z = a + b i$, where $\boxed{i^2 = -1}$.

Real part (a), Imaginary part (b)

Addition / Subtraction: $(a+bi) \pm (c+di) = (a \pm c) + (b \pm d)i$.

Multiplication: $(a+bi)(c+di) = (ac-bd) + (ad+bc)i$.

Division: The complex conjugate of $z = a+bi$ is $\bar{z} = a-bi$.

today. The modulus of $z$ is $|z| = {}^{+}\sqrt{z\bar{z}} = \sqrt{a^2+b^2}$.

What is $\frac{w}{z} = \frac{a+bi}{c+di}$ $(=$ some $x+yi)$ ?

Trick you should use: multiply $\frac{w}{z}$ by $\frac{\bar{z}}{\bar{z}} = 1$:

$$\frac{w}{z} \times \frac{\bar{z}}{\bar{z}} = \frac{w\bar{z}}{z\bar{z}} = \frac{w\bar{z}}{|z|^2} \Bigg\}$$ OK as long as $|z|^2 \neq 0$, $c^2+d^2 \neq 0$

i.e. atleast one of $c \neq 0$ & $d \neq 0$

i.e. $z \neq 0$.

$$= \frac{(a+bi)(c-di)}{c^2+d^2} = \underbrace{\boxed{\frac{(ac+bd)}{c^2+d^2}}}_{x} + \underbrace{\boxed{\frac{(bc-ad)}{c^2+d^2}}}_{y} i$$

Special case:

$$z^{-1} = \frac{1}{z} = \frac{1}{z}\left(\frac{\bar{z}}{\bar{z}}\right) = \frac{\bar{z}}{z\bar{z}} = \frac{\bar{z}}{|z|^2}$$ ← as long as $z \neq 0$.

**Example** Find $\dfrac{6+7i}{-3+2i}$.

**Solution** 2 routes: ① Trick $\dfrac{(6+7i)(-3-2i)}{(-3+2i)(-3-2i)}$

> **Notice**
> $|z| = |\bar{z}|$

$$= \frac{-18-21i-12i-14i^2}{(-3)^2+2^2}$$

$$= \frac{-4-33i}{13} = -\frac{4}{13} - \frac{33}{13}i.$$

② First find $\dfrac{1}{-3+2i} = \dfrac{\overline{(-3+2i)}}{(-3)^2+2^2} = \dfrac{-3-2i}{13} = -\dfrac{3}{13} - \dfrac{2}{13}i.$

Now $\dfrac{6+7i}{-3+2i} = (6+7i)\left(\dfrac{1}{-3+2i}\right) = (6+7i)\left(-\dfrac{3}{13} - \dfrac{2}{13}i\right)$

$$= -\frac{18}{3} - \frac{12}{3}i - \frac{21}{3}i - \frac{14}{3}i^2$$

$$= -\frac{4}{13} - \frac{33}{13}i.$$

## More Facts about Complex Conjugates

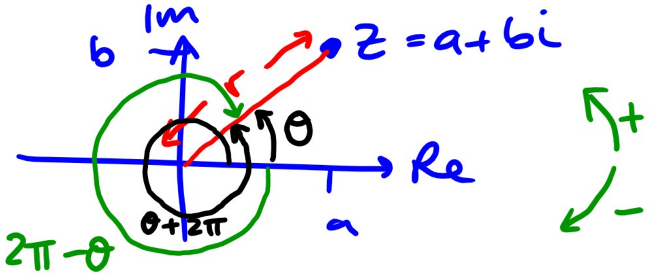
① $\bar{\bar{z}} = z$ ② $\overline{w \pm z} = \bar{w} \pm \bar{z}$

③ $\overline{wz} = \bar{w}\,\bar{z}$ ④ $\overline{\left(\dfrac{w}{z}\right)} = \dfrac{\bar{w}}{\bar{z}}$.

# Polar Form of Complex Numbers

We can represent $z = a+bi$ as $(a,b)$ in

Complex plane




Another way to plot $z$ is by giving:

radius $r = |z|$

an angle $\theta$ : any $\theta$ s.t. $a = r\cos\theta$
with the +ve real axis $b = r\sin\theta$

↪ called "the" argument of $z$, $\arg(z)$

( $\arg(z)$ not uniquely defined) ↪ For any argument $\theta$, $\theta + 2\pi k$ will also be an arg. for any integer $k$.

The principal argument of $z$, $\mathrm{Arg}(z)$, is the argument of $z$ in $(-\pi, \pi]$.

This gives us "the" polar form of $z = a+bi$

$$= r\cos\theta + (r\sin\theta)i$$

$$= r(\cos\theta + i\sin\theta)$$

**Example** Find the polar forms of (i) $z=-3-\sqrt{3}i$

(ii) $z=2i$.

**Solutions** (i) $r=|z|=\sqrt{(-3)^2+(-\sqrt{3})^2}=\sqrt{9+3}=\sqrt{12}$
$=2\sqrt{3}$.

$\theta$ satisfies $-3=r\cos\theta$
$=2\sqrt{3}\cos\theta \Rightarrow \cos\theta=\frac{-3}{2\sqrt{3}}=-\frac{\sqrt{3}}{2}$.

& $-\sqrt{3}=r\sin\theta$
$=2\sqrt{3}\sin\theta \Rightarrow \sin\theta=\frac{-\sqrt{3}}{2\sqrt{3}}=-\frac{1}{2}$.

Find one common solution $\theta$; all other solutions are $\theta+2\pi k$, $k$ integer.

e.g. $\theta=\ldots,-\frac{5\pi}{6}, \frac{7\pi}{6}, \ldots$ $\quad Arg(z)=-\frac{5\pi}{6}$

So polar form: $z=-3-\sqrt{3}i=2\sqrt{3}\left(\cos\left(-\frac{5\pi}{6}\right)+i\sin\left(-\frac{5\pi}{6}\right)\right)$

(ii) $z=2i$

$r=|z|=2$

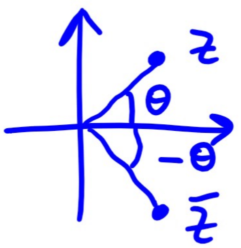
$\theta$ satisfies $0=2\cos\theta \Rightarrow \cos\theta=0$
$2=2\sin\theta \Rightarrow \sin\theta=1$

$Arg(z)=\frac{\pi}{2}\in(-\pi,\pi]$ so

polar form is $z=2\left(\cos\left(\frac{\pi}{2}\right)+i\sin\left(\frac{\pi}{2}\right)\right)$.

**Notice** If $z = r(\cos(\theta) + i\sin(\theta))$, then

$$\bar{z} = r(\cos(\theta) - i\sin(\theta))$$
$$= r(\cos(-\theta) + i\sin(-\theta))$$




args for $\bar{z}$ are the negative of the args for $z$.

## Multiplication & Division of complex #s in polar form

Things get easier!

It is possible using Maclaurin series to show:

$$\cos\theta + i\sin\theta = e^{i\theta}$$

You should know this Fact, even if you don't know where it comes from.

Say $z_1 = r_1(\cos\theta_1 + i\sin\theta_1)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Multiply: $z_1 z_2 = r_1 r_2 (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$

$$= r_1 r_2 e^{i\theta_1} e^{i\theta_2}$$

$$= r_1 r_2 e^{i(\theta_1 + \theta_2)}$$

also use double angle formulae

$$= r_1 r_2 (\cos(\theta_1 + \theta_2) + i\sin(\theta_1 + \theta_2)).$$

↳ so $|z_1 z_2| = |z_1||z_2|$

& "$\arg(z_1 z_2) = \arg(z_1) + \arg(z_2)$"

i.e. to get arg of $z_1 z_2$ add args of $z_1$ & $z_2$.

For Division, $\dfrac{z_1}{z_2} = \dfrac{r_1 e^{i\theta_1}}{r_2 e^{i\theta_2}} = \left(\dfrac{r_1}{r_2}\right) e^{i\theta_1 - i\theta_2}$

(as long as $z_2 \neq 0$)

$= \dfrac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$

$= \dfrac{r_1}{r_2}(\cos(\theta_1-\theta_2)+i\sin(\theta_1-\theta_2))$

So to get $\left|\dfrac{z_1}{z_2}\right| = \dfrac{|z_1|}{|z_2|}$ & to get arg of $\dfrac{z_1}{z_2}$ subtract arg of $z_2$ from arg of $z_1$.