1B03 – LINEAR ALGEBRA 1 (C01) WS19 — Lecture 2

Last time: SYSTEMS OF LINEAR EQUATIONS

m equations in n variables

$$\begin{cases} (1)\; a_{11}x_1 + a_{12}x_2 + \dots + a_{1n}x_n = b_1 \\ (2)\; a_{21}x_1 + a_{22}x_2 + \dots + a_{2n}x_n = b_2 \\ \vdots \\ (m)\; a_{m1}x_1 + a_{m2}x_2 + \dots + a_{mn}x_n = b_m \end{cases}$$

a_{ij} = coefficient in equation #i of variable x_j

A solution is an n-tuple solving all m equations at once.

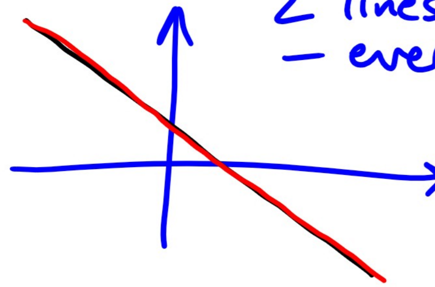
We can represent a system of L.E.s using a matrix (rectangular array of #s). More precisely, the augmented matrix of the system above is:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right]$$

↑ line optional

Back to lines: the system $\left\{\begin{array}{l} x + 3y = 6 \\ 2x - y = 1 \end{array}\right\}$ had $\underline{\underline{1}}$ solution

Example $\left\{ \begin{array}{l} x + y = 5 \\ 3x + 3y = 15 \end{array} \right\}$



2 lines coincide – every point is a solution: $\underline{\underline{\infty}}$-many solutions

Our usual strategy:

$$3x + 3y - 3(x+y) = 15 - 3.5$$

$$0 = 0$$

Now have $\left\{ \begin{array}{l} x + y = 5 \\ 0 = 0 \end{array} \right\}$

So having a 2nd equation gives no extra info.

(x,y) with $y = 5 - x$

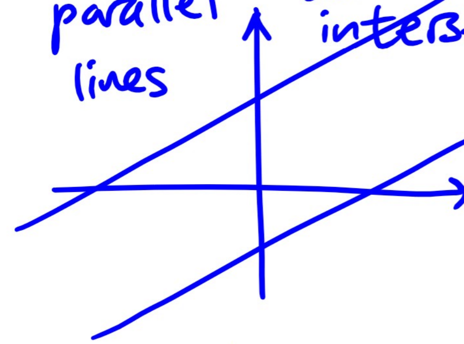
We can write this using a <u>parameter</u>, t say:

Solutions are $(t, 5-t)$

e.g. $(0,5), (\pi, 5-\pi), (-\sqrt{3}, 5+\sqrt{3})$

Example $\left\{ \begin{array}{l} x - 2y = -10 \\ 2x - 4y = 6 \end{array} \right\}$

parallel lines – don't intersect



Usual strategy:

$$2x - 4y - 2(x - 2y) = 6 - 2(-10)$$

$$0 = 26$$

nonsense!

$\underline{\underline{0}}$ solutions

In fact these cover all possibilities for any system of L.E.s — we will show that every system of L.E.s has either

- 0 solutions $\longrightarrow$ inconsistent system
- or 1 solution
- or ∞-many solutions

$\left.\begin{array}{l} \text{1 solution} \\ \infty\text{-many solutions} \end{array}\right\} \rightarrow$ consistent system

A more complicated example — follow the matrix representation.

Example Solve

Augmented matrix:

$$\begin{array}{c} R_1 \\ R_2 \\ R_3 \end{array} \begin{bmatrix} 1 & -3 & 1 & 1 \\ -2 & 1 & -2 & 1 \\ 2 & -1 & -1 & 1 \end{bmatrix}$$

(1) $x - 3y + z = 1$
(2) $-2x + y - 2z = 1$
(3) $2x - y - z = 1$

(3) $\to$ (2) + (3) to elim. x & y:

$R_3 \to R_2 + R_3$

(1) $x - 3y + z = 1$
(2) $-2x + y - 2z = 1$
(4) $-3z = 2$

$$\begin{bmatrix} 1 & -3 & 1 & 1 \\ -2 & 1 & -2 & 1 \\ 0 & 0 & -3 & 2 \end{bmatrix}$$

(4) $\to -\frac{1}{3}$(4) to solve for z:

$R_3 \to -\frac{1}{3} R_3$

(1) $x - 3y + z = 1$
(2) $-2x + y - 2z = 1$
(5) $z = -2/3$

$$\begin{bmatrix} 1 & -3 & 1 & 1 \\ -2 & 1 & -2 & 1 \\ 0 & 0 & 1 & -2/3 \end{bmatrix}$$

(2) $\to$ 2(1) + (2)

$R_2 \to 2R_1 + R_2$

(1) $x - 3y + z = 1$
(6) $-5y = 3$
(5) $z = -2/3$

$$\begin{bmatrix} 1 & -3 & 1 & 1 \\ 0 & -5 & 0 & 3 \\ 0 & 0 & 1 & -2/3 \end{bmatrix}$$

(6) $\to -\frac{1}{5}$(6) to solve for y:

$R_2 \to -\frac{1}{5} R_2$

(1) $x - 3y + z = 1$

(7) $y = -3/5$

(5) $z = -2/3$

$$\begin{bmatrix} 1 & -3 & 1 & 1 \\ 0 & 1 & 0 & -3/5 \\ 0 & 0 & 1 & -2/3 \end{bmatrix}$$

$$\Big\downarrow R_1 \to R_1 + 3R_2 - R_3$$

(1) $\to$ (1) $+ 3$(7) $-$ (5)

(8) $x = 1 - \frac{9}{5} + \frac{2}{3} = -\frac{2}{15}$

(7) $y = -3/5$

(5) $z = -2/3$

$$\begin{bmatrix} 1 & 0 & 0 & -2/15 \\ 0 & 1 & 0 & -3/5 \\ 0 & 0 & 1 & -2/3 \end{bmatrix} \leftarrow A$$

We can read off the solution:

$$(x, y, z) = \left(-\frac{2}{15}, -\frac{3}{5}, -\frac{2}{3}\right)$$

Earlier examples from matrix point of view:

$x + y = 5$
$3x + 3y = 15$

$$\begin{bmatrix} 1 & 1 & 5 \\ 3 & 3 & 15 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 3R_1} \begin{bmatrix} 1 & 1 & 5 \\ 0 & 0 & 0 \end{bmatrix} = B$$

$x - 2y = -10$
$2x - 4y = 6$

$$\begin{bmatrix} 1 & -2 & -10 \\ 2 & -4 & 6 \end{bmatrix} \xrightarrow{R_2 \to R_2 - 2R_1} \begin{bmatrix} 1 & -2 & -10 \\ 0 & 0 & 26 \end{bmatrix}$$

$$\Big\downarrow R_2 \to \frac{1}{26} R_2$$

$$C = \begin{bmatrix} 1 & -2 & -10 \\ 0 & 0 & 1 \end{bmatrix}$$

Important Properties All of the end matrices (A, B, C) above satisfy:

(1) In every row, the "leading entry" i.e. the left-most non-zero entry, is a 1

(2) All zero rows are at the bottom.

(3) Every "leading 1" (as in (1)) in a non-zero row is to the right of all leading 1s in rows higher up.

A, B also satisfy:

(4) A leading 1 is the only non-zero entry in its column.

Definitions (REF) A matrix satisfying (1) - (3) is said to be in row echelon form

(RREF) A matrix satisfying (1) - (4) is said to be in reduced row echelon form.