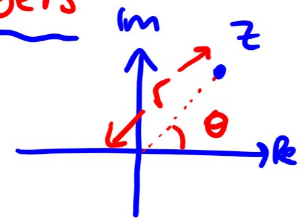
1B03 – LINEAR ALGEBRA 1 (C01) WS19 — Lecture 20

Yesterday Polar Form of Complex Numbers



$$z = a + ib = r(\cos\theta + i\sin\theta) = re^{i\theta}$$

- radius $r = |z|$
- argument θ is any angle satisfying $a = r\cos\theta$, $b = r\sin\theta$.

[Principal argument: $\theta \in (-\pi, \pi]$.]

Multiplication / Division:

$$\big(r_1(\cos\theta_1 + i\sin\theta_1)\big)\big(r_2(\cos\theta_2 + i\sin\theta_2)\big) = r_1r_2\big(\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)\big)$$

[To multiply: multiply radii, add arguments.]

$$\big(r_1(\cos\theta_1 + i\sin\theta_1)\big) \Big/ \big(r_2(\cos\theta_2 + i\sin\theta_2)\big) = \frac{r_1}{r_2}\big(\cos(\theta_1-\theta_2) + i\sin(\theta_1-\theta_2)\big)$$

[To divide: divide radii, subtract arguments.]

Examples Write (i) $(2i)(-3-\sqrt{3}i)$ (ii) $\dfrac{2i}{-3-\sqrt{3}i}$ in polar form.

Yesterday: $2i = 2\left(\cos\left(\frac{\pi}{2}\right) + i\sin\left(\frac{\pi}{2}\right)\right)$ ← $r = 2$, $\theta = \pi/2$

$-3-\sqrt{3}i = 2\sqrt{3}\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$. ← $r = 2\sqrt{3}$, $\theta = -\frac{5\pi}{6}$

Solution (i) $(2i)(-3-\sqrt{3}i) = 4\sqrt{3}\left(\cos\left(\underbrace{\frac{\pi}{2} - \frac{5\pi}{6}}_{\pi/2 + (-\frac{5\pi}{6})}\right) + i\sin\left(\frac{\pi}{2} - \frac{5\pi}{6}\right)\right)$

$= 4\sqrt{3}\left(\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right)$

(ii) $\dfrac{2i}{-3-\sqrt{3}i} = \dfrac{2}{2\sqrt{3}}\left(\cos\left(\frac{\pi}{2} - \left(-\frac{5\pi}{6}\right)\right) + i\sin\left(\frac{\pi}{2} - \left(-\frac{5\pi}{6}\right)\right)\right)$

$= \dfrac{2}{2\sqrt{3}}\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right)$ OK.

$$= \frac{2}{2\sqrt{3}}\left(\cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)\right)$$

↖ Principal arg. in $(-\pi, \pi]$

In general $z^n = (re^{i\theta})^n = r^n e^{in\theta}$

$$= r^n \boxed{(\cos(n\theta) + i\sin(n\theta))}$$

$$\overset{\shortparallel}{\left[r(\cos(\theta) + i\sin\theta)\right]^n}$$

$$\overset{\shortparallel}{r^n} \boxed{(\cos(\theta) + i\sin(\theta))^n}$$

De Moivre's Rule

Roots — Want to find $z_1^{1/n}$ (nth root of z_1) for $n \neq 0$ integer

i.e. want to find z_2 with $z_2^n = z_1$

i.e. want $\underset{|z_2|}{r_2}$, $\underset{\arg(z_2)}{\theta_2}$ with $\underbrace{r_2^n e^{in\theta_2}}_{z_2^n} = \underbrace{r_1 e^{i\theta_1}}_{z_1}$

So $r_2^n = r_1 \Rightarrow r_2 = \sqrt[n]{r_1}$ ← real #

& we need to find all θ_2 with $n\theta_2$ some argument of z_1.

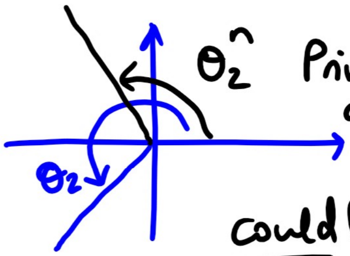
we can write $\theta_1 = \underset{\uparrow \atop \mathrm{Arg}(z_1)}{\alpha_1} + 2\pi k$, k integer

Now $n\theta_2 = \alpha_1 + 2\pi k$, any integer k,

$\Rightarrow \boxed{\theta_2 = \frac{\alpha_1}{n} + \frac{2\pi k}{n}}$ for any integer k.

$\uparrow$ Enough to work out θ_2 for $k = 0, \dots, n-1$.

Caution

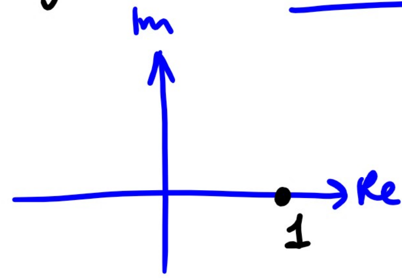


θ_2^n Principal arg. of θ_2^n could be smaller than (principal arg. of) θ_2.

Example Find the cube roots of 1.
$\hookrightarrow n = 3$

Solution $r = |1| = 1$

$\arg(1) = 0$



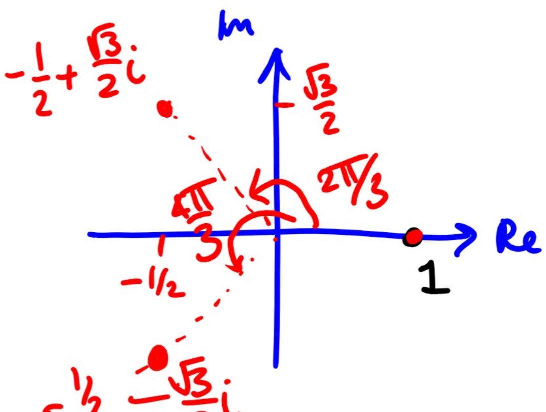
So cube roots z of 1 $\boxed{z^3 = 1}$ have $|z| = \sqrt[3]{1} = 1$

& $\arg(z) = \dfrac{0}{3} + \dfrac{2\pi k}{3}$, for $k = 0, 1, 2$.

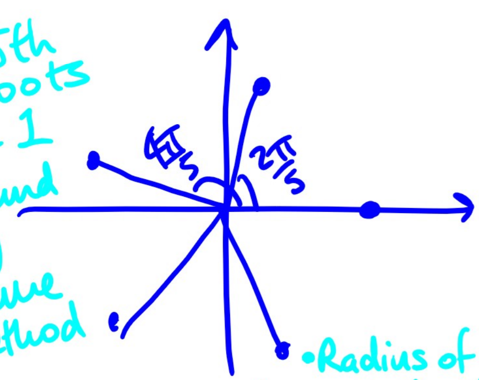
So cube roots of 1 are $(k=0)$: $1(\cos(0) + i\sin(0)) = 1$

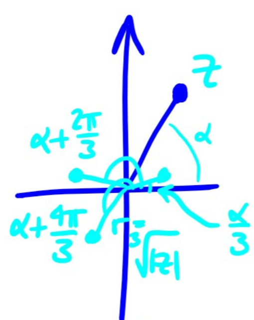
$(k=1)$: $1\left(\cos\left(\frac{2\pi}{3}\right) + i\sin\left(\frac{2\pi}{3}\right)\right) = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$

$(k=2)$: $1\left(\cos\left(\frac{4\pi}{3}\right) + i\sin\left(\frac{4\pi}{3}\right)\right) = -\frac{1}{2} - \frac{\sqrt{3}}{2}i$



5th roots of 1 found by same method





- Radius of all nth roots of z is $|z|^{\frac{1}{n}}$
- Args are $\frac{\alpha}{n}, \frac{\alpha}{n} + \frac{2\pi}{n}, \frac{\alpha}{n} + \frac{4\pi}{n}, \dots, \frac{\alpha}{n} + \frac{2(n-1)\pi}{n}$ $(\alpha = \operatorname{Arg}(z))$

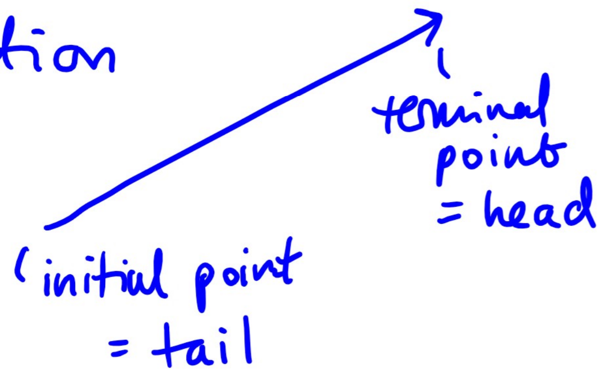
3.1 Vectors in 2-space, 3-space, n-space

Back to the real world (for now...)

Vectors are • matrices $1 \times n$ (row vector)

$m \times 1$ (column vector)

- in $\mathbb{R}^2$ & $\mathbb{R}^3$, line segments with magnitude & direction



↳ location irrelevant i.e. vectors <u>same</u> if they have same magnitude & direction

↳ encode line segment by placing tail at origin & representing the vector using coordinates (x, y) or (x, y, z) of location of head.

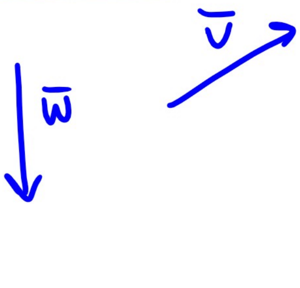
This is then the row vector $\begin{bmatrix} x & y \end{bmatrix}$ or column vector $\begin{bmatrix} x \\ y \end{bmatrix}$ with components x & y.

Also the 2-tuple (x, y)

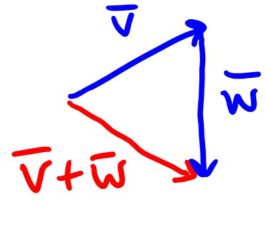
Arithmetic then all works!

i.e. using this encoding we can translate pictures of vector operations into natural arithmetic operations of 2-tuples & 3-tuples

- <u>add</u> vectors

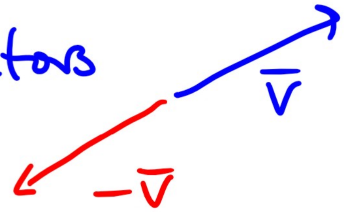


$\bar{v} + \bar{w}$ is given by



e.g. $(2,3) + (0,-5) = (2,-2)$

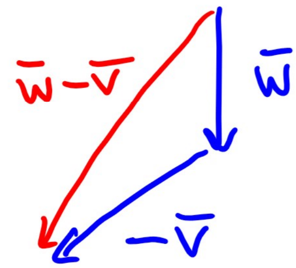
- <u>negate</u> vectors



$-\bar{v}$ is the vector with same magnitude as $\bar{v}$ & opposite direction

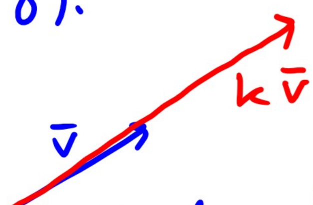
e.g. $-(2,3) = (-2,-3)$.

- <u>subtract</u> vectors : $\bar{w} - \bar{v}$



e.g. $(0,-5) - (2,3)$
$= (-2,-8)$.

- <u>scale</u> vectors : $k\bar{v}$



has magnitude equal to magnitude of $\bar{v} \times k$
& same direction as $\bar{v}$ if $k > 0$
opposite " " " if $k < 0$

e.g. $2(2,3) = (4,6)$

We can translate Pictures $\longrightarrow$ into component-wise operations.

The point: we can extend all ideas with vectors as line segments (operations, magnitude, ...) from $\mathbb{R}^2$ (2-space) & $\mathbb{R}^3$ (3-space) to ... $\mathbb{R}^n$ (n-space).
(even though we cannot visualize them any more).