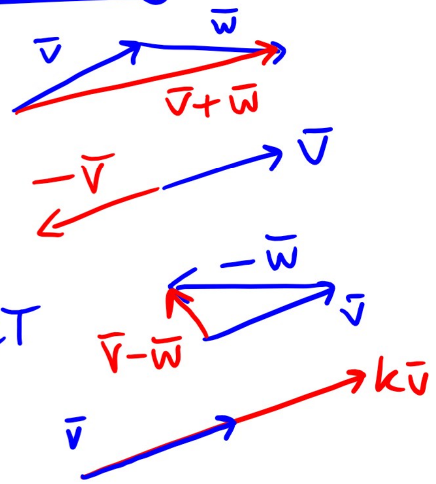
1B03 – LINEAR ALGEBRA 1 (C01) WS19 Lecture 21

Last Time Vectors in 2-space / 3-space, ...

We equate line segments with 2-tuples / 3-tuples.

Line Segments	Component-wise Arithmetic
• ADD	• e.g. $(5,-3)+(4,1)=(9,-2)$
• NEGATE	• e.g. $-(-3,7)=(3,-7)$
• SUBTRACT	• e.g. $(5,-3)-(4,1)=(1,-4)$
• SCALE	• e.g. $-3(10,1)=(-30,-3)$.



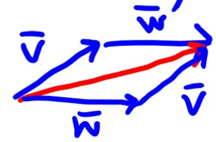
Goal Extend all these ideas & more to $\mathbb{R}^n$ (n-space) using the encoding of vectors by n-tuples $(x_1, \ldots, x_n)$ (or $[x_1 \cdots x_n]$ or $\begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$)

Basic Operations If $\bar{v}=(v_1,\ldots,v_n)$ and $\bar{w}=(w_1,\ldots,w_n)$ in $\mathbb{R}^n$, then

- $\bar{v}+\bar{w} = (v_1+w_1, v_2+w_2, \ldots, v_n+w_n)$
- $-\bar{v} = (-v_1, -v_2, \ldots, -v_n)$
- $\bar{w}-\bar{v} = (w_1-v_1, w_2-v_2, \ldots, w_n-v_n)$
- $k\bar{v} = (kv_1, kv_2, \ldots, kv_n)$ for $k \in \mathbb{R}$
- $\bar{0} = (0, \ldots, 0)$ called the <u>zero vector</u>. ($= 0\bar{u}$ for any $\bar{u}$) scalar

Facts If $\bar{v}, \bar{w}, \bar{u} \in \mathbb{R}^n$, $k \in \mathbb{R}$, then

- $\bar{v} + \bar{w} = \bar{w} + \bar{v}$ in $\mathbb{R}^2$



- $(\bar{v} + \bar{w}) + \bar{u} = \bar{v} + (\bar{w} + \bar{u})$

So order of addition of vectors doesn't matter

- $k(\bar{v} + \bar{w}) = k\bar{v} + k\bar{w}$
- $\bar{v} + (-\bar{v}) = \bar{0}$
- $\bar{v} + \bar{0} = \bar{v}$
- $1\bar{v} = \bar{v}$

For more see textbook Theorem 3.1.1.

Definition A vector $\bar{w} \in \mathbb{R}^n$ is a <u>linear combination</u> of $\bar{v}_1, \bar{v}_2, \ldots, \bar{v}_r \in \mathbb{R}^n$ if there are scalars $\underbrace{k_1, \ldots, k_r}$ $\in \mathbb{R}$ with

$$\bar{w} = k_1\bar{v}_1 + k_2\bar{v}_2 + \ldots + k_r\bar{v}_r .$$

called <u>coefficients</u>

Example Find scalars k_1, k_2, k_3 such that

$$(2, -1, 3) = k_1(3, 1, -1) + k_2(-1, 6, 2) + k_3(2, 1, 1).$$

Solution

$$= (3k_1, k_1, -k_1) + (-k_2, 6k_2, 2k_2) + (2k_3, k_3, k_3)$$

$$= (3k_1 - k_2 + 2k_3,\ k_1 + 6k_2 + k_3,\ -k_1 + 2k_2 + k_3)$$

($\bar{v} = \bar{w}$ iff $v_i = w_i$ for all $i = 1, \ldots, n$)

So need to solve
$$\begin{aligned} 3k_1 - k_2 + 2k_3 &= 2 \\ k_1 + 6k_2 + k_3 &= -1 \\ -k_1 + 2k_2 + k_3 &= 3 \end{aligned}$$

i.e. need to row reduce augmented matrix:

$$\left[\begin{array}{ccc|c} 3 & -1 & 2 & 2 \\ 1 & 6 & 1 & -1 \\ -1 & 2 & 1 & 3 \end{array}\right]$$
$$\underbrace{\bar{v}_1 \quad \bar{v}_2 \quad \bar{v}_3}_{} \qquad \bar{w}$$

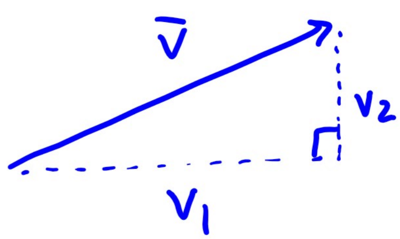
The question was: Find k_1, k_2, k_3 s.t.
$$\bar{w} = k_1\bar{v}_1 + k_2\bar{v}_2 + k_3\bar{v}_3$$
So we can form the augmented matrix directly from this equation

$$\left[\begin{array}{ccc|c} 1 & 0 & 0 & -6/5 \\ 0 & 1 & 0 & -2/5 \\ 0 & 0 & 1 & 13/5 \end{array}\right] \rightarrow \text{i.e. } k_1 = -6/5,\ k_2 = -2/5,\ k_3 = 13/5.$$

Check: $(2,-1,3) = -\frac{6}{5}\bar{v}_1 - \frac{2}{5}\bar{v}_2 + \frac{13}{5}\bar{v}_3$.

3.2 Norm, Dot Product & Distance in $\mathbb{R}^n$

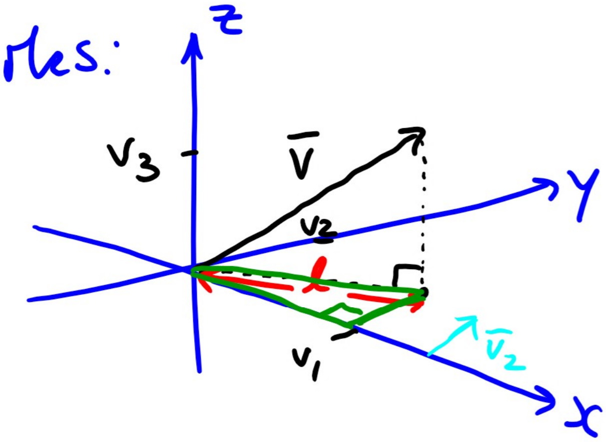
In $\mathbb{R}^2$



the magnitude of a vector $\bar{v} = (v_1, v_2)$ (also called length of $\bar{v}$ or the norm of $\bar{v}$) is

$$\|\bar{v}\| = \sqrt{v_1^2 + v_2^2}$$

In $\mathbb{R}^3$, Pythagoras also works:



$$\|\bar{v}\|^2 = l^2 + v_3^2$$
$$= (v_1^2 + v_2^2) + v_3^2$$

$$\|\bar{v}\| = \sqrt{v_1^2 + v_2^2 + v_3^2}$$

__Definition__ In $\mathbb{R}^n$ the __norm__ (= __magnitude__ = __length__ = etc.) of a vector $\bar{v} = (v_1, v_2, \ldots, v_n)$ is $\|\bar{v}\| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}$.

__Example__ $\|(-7, 5, 2, -1)\| = \sqrt{(-7)^2 + 5^2 + 2^2 + (-1)^2}$
$$= \sqrt{49 + 25 + 4 + 1}$$
$$= \sqrt{79}.$$

__Facts about Norms__

For any $\bar{v} \in \mathbb{R}^n$:

(1) $\|\bar{v}\| \geq 0$

(2) $\|\bar{v}\| = 0$ exactly when $\bar{v} = \bar{0}$

(3) $\|k\bar{v}\| = |k| \, \|\bar{v}\|$, $k \in \mathbb{R}$

__Definition__ If $\|\bar{v}\| = 1$, we call $\bar{v}$ a __unit vector__. If $\bar{w} \neq \bar{0}$ we can

<u>normalize</u> $\overline{w}$ (turn $\overline{w}$ into a unit vector of same direction as $\overline{w}$) : $\dfrac{1}{\|\overline{w}\|}\overline{w}$ is a unit vector

written $\dfrac{\overline{w}}{\|\overline{w}\|}$

<u>Special case</u> <u>Standard Unit Vectors in $\mathbb{R}^n$</u>

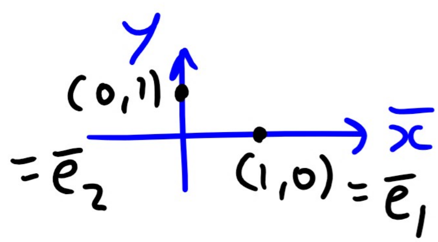
$\overline{e}_1 = (1, 0, \ldots, 0)$

$\overline{e}_2 = (0, 1, 0, \ldots, 0)$

$\overline{e}_3 = (0, 0, 1, 0, \ldots, 0)$

$\ldots$ $\overline{e}_n = (0, \ldots, 0, 1)$

In $\mathbb{R}^2$, $\mathbb{R}^3$, these are the unit vectors along the coordinate axes:



<u>Key Fact</u>

Every vector $\overline{v} = (v_1, v_2, \ldots, v_n)$ in $\mathbb{R}^n$ can be written as a linear combination of $\overline{e}_1, \ldots, \overline{e}_n$:

$$\overline{v} = v_1\overline{e}_1 + v_2\overline{e}_2 + \ldots + v_n\overline{e}_n .$$