1B03 – LINEAR ALGEBRA 1 (C01) WS19 Lecture 22

Last Time

NORMS IN $\mathbb{R}^n$

The norm (= magnitude = length) of a vector

$\bar{v} = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$ is $\|\bar{v}\| = \sqrt{v_1^2 + v_2^2 + \dots + v_n^2}$.

① $\|\bar{v}\| \geq 0$ always.

② $\|\bar{v}\| = 0$ if & only if $\bar{v} = \bar{0}$.

③ $\|k\bar{v}\| = |k|\,\|\bar{v}\|$.

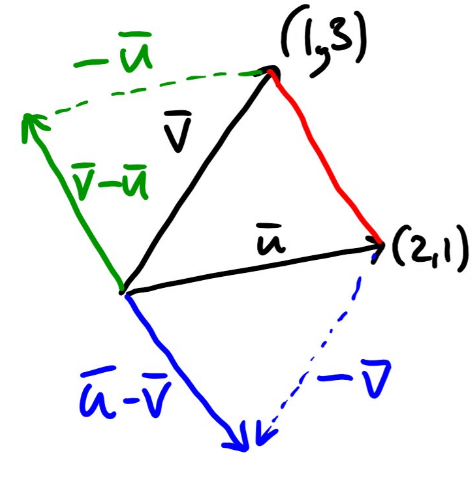
Unit vectors: any $\bar{v}$ with $\|\bar{v}\| = 1$ e.g. $\underbrace{\frac{1}{\|\bar{w}\|}}_{\text{normalization of } \bar{w}} \bar{w}$ for any vector $\bar{w}$

e.g. $\bar{e}_1 = (1, 0, \dots, 0)$, $\dots$, $\bar{e}_n = (0, \dots, 0, 1)$ ← standard unit vectors.

$$\bar{v} = (v_1, \dots, v_n) = v_1 \bar{e}_1 + \dots + v_n \bar{e}_n$$

Example $(3, 5, -27, 6) = 3(1,0,0,0) + 5(0,1,0,0) - 27(0,0,1,0) + 6(0,0,0,1)$

Distance between Vectors



Example Find the distance between $\bar{v} = (1, 3)$ & $\bar{u} = (2, 1)$

Distance $= \|\bar{u} - \bar{v}\| = \|\bar{v} - \bar{u}\|$

$= \|(2,1) - (1,3)\| = \|(1, -2)\| = \sqrt{1^2 + (-2)^2} = \sqrt{5}$.

In $\mathbb{R}^n$ we define the distance between $\bar{u}$ & $\bar{v}$

as $d(\bar{u}, \bar{v}) = \|\bar{u} - \bar{v}\| \quad (= \|\bar{v} - \bar{u}\|)$

(Careful: the $\bar{u}, \bar{v}$ here must be in same $\mathbb{R}^n$!)

The dot product

In $\mathbb{R}^2, \mathbb{R}^3$ based on angle θ between $\bar{u}$ & $\bar{v}$

$$\bar{u} \cdot \bar{v} = \|\bar{u}\| \|\bar{v}\| \cos\theta$$

In practice, to compute $\bar{u} \cdot \bar{v}$, we use

$$\bar{u} \cdot \bar{v} = \begin{cases} u_1 v_1 + u_2 v_2 & (\text{in } \mathbb{R}^2) \\ u_1 v_1 + u_2 v_2 + u_3 v_3 & (\text{in } \mathbb{R}^3) \end{cases}$$

(Comes from Law of Cosines)

So (for now) forget angles:

Again, $\bar{u}, \bar{v}$ must be in same $\mathbb{R}^n$ to take dot product.

Def^n In $\mathbb{R}^n$, $\bar{u} \cdot \bar{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$.

Example $(3, 0, 5, -1, 0) \cdot (2, 1, 0, 5, 3)$

$$= 6 + 0 + 0 - 5 + 0 = 1.$$

So for matrix multiplication: all just dot products!

Example: If $\bar{u}, \bar{v}$ are row vectors, then

$\bar{u} = [u_1 \cdots u_n]$, $\bar{v} = [v_1 \cdots v_n]$

$$\bar{u} \cdot \bar{v} = \bar{u}\bar{v}^T = \bar{v}\bar{u}^T$$

Similar if $\bar{u}$ and/or $\bar{v}$ are column vectors

(see p. 151)

<u>Notice</u> For $\bar{v} \in \mathbb{R}^n$

$$\bar{v} \cdot \bar{v} = v_1 v_1 + v_2 v_2 + \ldots + v_n v_n$$
$$= v_1^2 + \ldots + v_n^2$$
$$= \|\bar{v}\|^2$$

So $\boxed{\|\bar{v}\| = \sqrt{\bar{v} \cdot \bar{v}}}$

<u>Example</u> Suppose $\|\bar{u}\| = 2$, $\|\bar{v}\| = 3$ & $\bar{u} \cdot \bar{v} = -6$.

Find $d(\bar{u}, \bar{v})$.

<u>Solution</u> $d(\bar{u}, \bar{v}) = \|\bar{u} - \bar{v}\|$

$$= \sqrt{(\bar{u} - \bar{v}) \cdot (\bar{u} - \bar{v})}$$

$$= \sqrt{\bar{u} \cdot \bar{u} - \bar{v} \cdot \bar{u} - \bar{u} \cdot \bar{v} + \bar{v} \cdot \bar{v}}$$

$$= \sqrt{\|\bar{u}\|^2 - 2\bar{u} \cdot \bar{v} + \|\bar{v}\|^2}$$

$$= \sqrt{2^2 - 2(-6) + 3^2} = \sqrt{4 + 12 + 9} = 5.$$

Dot products in $\mathbb{R}^n$ behave just like regular multiplication of real #s.

Recall in $\mathbb{R}^2, \mathbb{R}^3$: $\bar{u} \cdot \bar{v} = \|\bar{u}\| \|\bar{v}\| \cos\theta$, θ : angle between $\bar{u}, \bar{v}$

i.e. $\theta = \cos^{-1}\left(\dfrac{\bar{u} \cdot \bar{v}}{\|\bar{u}\|\|\bar{v}\|}\right)$

In $\mathbb{R}^n$ <u>define</u> angle between $\bar{u}$ & $\bar{v}$ to be this θ

Careful : is $\dfrac{\bar{u} \cdot \bar{v}}{\|\bar{u}\|\|\bar{v}|} \in [-1, 1]$? Yes :

<u>Cauchy - Schwarz Inequality</u> - for $\bar{u}, \bar{v}$ in $\mathbb{R}^n$.

$$|\bar{u} \cdot \bar{v}| \leq \|\bar{u}\| \|\bar{v}\|$$

<u>Example</u> Find the angle between $\bar{u} = (1, 1, 0, 2)$ & $\bar{v} = (-1, 0, 0, -1)$.

<u>Solution</u> Find $\bar{u} \cdot \bar{v} = -1 + 0 + 0 - 2 = -3$

& $\|\bar{u}\| = \sqrt{1^2 + 1^2 + 2^2} = \sqrt{6} = \sqrt{2}\sqrt{3}$

& $\|\bar{v}\| = \sqrt{(-1)^2 + (-1)^2} = \sqrt{2}$.

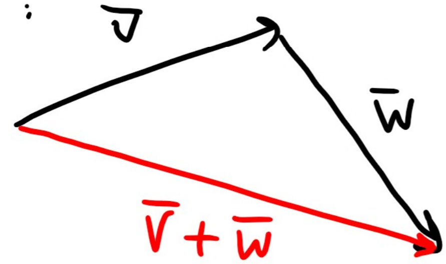
So $\cos(\theta) = \dfrac{\bar{u} \cdot \bar{v}}{\|\bar{u}\|\|\bar{v}\|} = \dfrac{-3}{\sqrt{2}\sqrt{3}\sqrt{2}} = -\dfrac{\sqrt{3}}{2}$.

$\theta = 5\pi/6$.

For all $\bar{v}, \bar{w} \in \mathbb{R}^n$ we have:

1 Triangle Inequality

In $\mathbb{R}^2$:

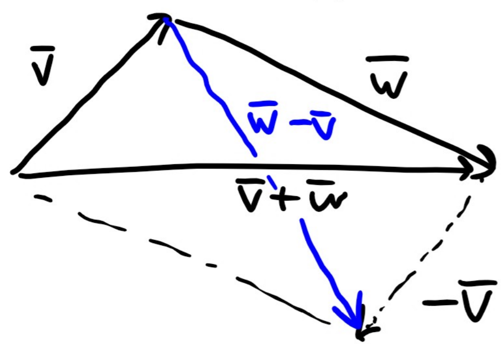


$$\|\bar{v}+\bar{w}\| \leq \|\bar{v}\| + \|\bar{w}\|$$

– see textbook for proof

2 Parallelogram Equation

In $\mathbb{R}^2$:



$$\|\bar{v}+\bar{w}\|^2 + \|\bar{w}-\bar{v}\|^2 = 2\left(\|\bar{v}\|^2 + \|\bar{w}\|^2\right)$$

$$(\bar{v}+\bar{w})\cdot(\bar{v}+\bar{w}) + (\bar{w}-\bar{v})\cdot(\bar{w}-\bar{v})$$

$$= \|\bar{v}\|^2 + \cancel{2\bar{v}\cdot\bar{w}} + \|\bar{w}\|^2 + \|\bar{w}\|^2 - \cancel{2\bar{w}\cdot\bar{v}} + \|\bar{v}\|^2$$

$$= 2\left(\|\bar{v}\|^2 + \|\bar{w}\|^2\right)$$

3

$$\bar{v}\cdot\bar{w} = \frac{1}{4}\|\bar{v}+\bar{w}\|^2 - \frac{1}{4}\|\bar{w}-\bar{v}\|^2$$

Try as an Exercise first, then see textbook p. 150.