1B03 – LINEAR ALGEBRA 1 (C01) WS19 — Lecture 23

Yesterday

Dot Product & Angle

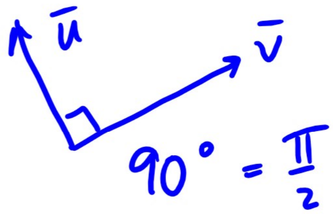
For $\bar{u}, \bar{v}$ in $\mathbb{R}^n$, $\bar{u} \cdot \bar{v} = \|\bar{u}\| \|\bar{v}\| \cos(\theta)$

this formula now defines the angle between $\bar{u}$ & $\bar{v}$

while $\bar{u} \cdot \bar{v} = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$ defines the dot product

Facts $*$ $\|\bar{u}\| = \sqrt{\bar{u} \cdot \bar{u}}$; $*$ $\cdot$ behaves like multiplication of reals;
$*$ $d(\bar{u}, \bar{v}) = \|\bar{u} - \bar{v}\|$; $*$ $|\bar{u} \cdot \bar{v}| \leq \|\bar{u}\| \|\bar{v}\|$; $*$ $\|\bar{u} + \bar{v}\| \leq \|\bar{u}\| + \|\bar{v}\|$.

3.3 Orthogonality $\rightarrow$ generalizes "perpendicular"



$\bar{u} \cdot \bar{v} = \|\bar{u}\| \|\bar{v}\| \cos(\frac{\pi}{2}) = 0$

Def^n $\bar{u}, \bar{v} \in \mathbb{R}^n$. $\bar{u}$ & $\bar{v}$ are orthogonal if $\bar{u} \cdot \bar{v} = 0$.

Example $\bar{u} = (1, -1, 3, 2)$, $\bar{v} = (7, -5, -2, -3)$

$\bar{u} \cdot \bar{v} = 7 + 5 - 6 - 6 = 0$

Important Example in $\mathbb{R}^n$: standard unit vectors $\bar{e}_1, \dots, \bar{e}_n$

Any 2 orthogonal to one another $\bar{e}_i \cdot \bar{e}_j = 0$. $i \neq j$
("mutually orthogonal")

Pythagoras in $\mathbb{R}^n$

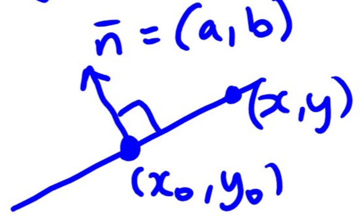
If $\bar{u}, \bar{v} \in \mathbb{R}^n$ orthogonal, then

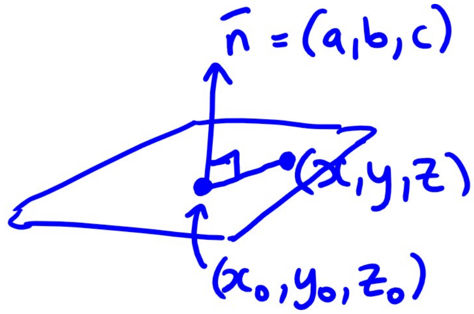
$$\|\bar{u}+\bar{v}\|^2 = \|\bar{u}\|^2 + \|\bar{v}\|^2$$

(Check this & ask me!)

Lines in $\mathbb{R}^2$ & Planes in $\mathbb{R}^3$

→ described uniquely with one point on line/plane & a normal vector $\bar{n}$ orthogonal to the line plane





Vector in line:

$(x-x_0, y-y_0)$

So $\bar{n}\cdot(x-x_0,y-y_0)=0$

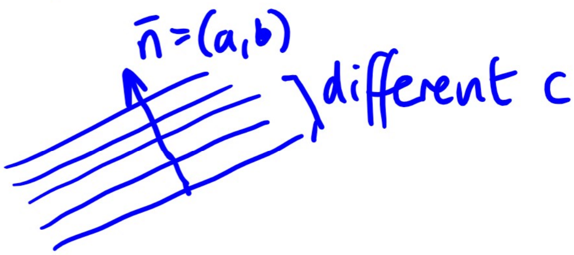
$\Rightarrow (a,b)\cdot(x-x_0,y-y_0)=0$

$\Rightarrow ax+by+(-ax_0-by_0)=0$

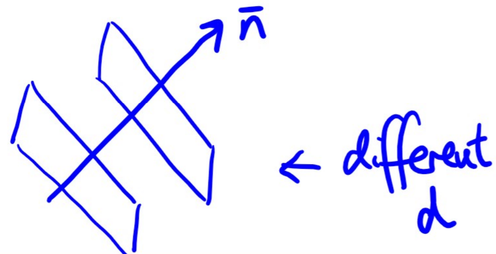
$(a,b,c)\cdot(x-x_0,y-y_0,z-z_0)=0$

$\Rightarrow ax+by+cz+(-ax_0-by_0-cz_0)=0$

So $ax+by+c=0$ is a line in $\mathbb{R}^2$ with normal vector $\bar{n}=(a,b)$



So $ax+by+cz+d=0$ is a plane in $\mathbb{R}^3$ with normal $\bar{n}=(a,b,c)$

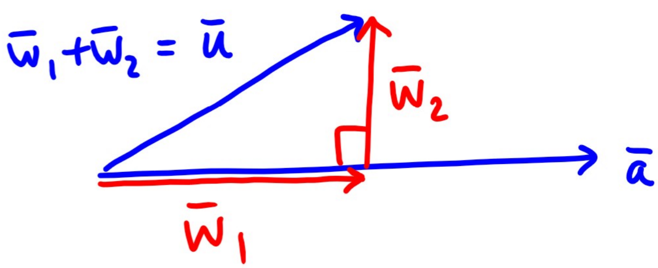


Projection Theorem

If $\bar{u}, \bar{a} \in \mathbb{R}^n$, $\bar{a} \neq \bar{0}$, then $\bar{u} = \bar{w}_1 + \bar{w}_2$ where

$\bar{w}_1 = k\bar{a}$ (some k to be determined) & $\bar{w}_2 \cdot \bar{a} = 0$

In $\mathbb{R}^2$



i.e. $\bar{u} = k \cdot \bar{a} + \bar{w}_2$

$$\text{So } \bar{u} \cdot \bar{a} = k\,\bar{a} \cdot \bar{a} + \overset{0}{\cancel{\bar{w}_2 \cdot \bar{a}}}$$

$$= k \|\bar{a}\|^2$$

$$\Rightarrow k = \frac{\bar{u} \cdot \bar{a}}{\|\bar{a}\|^2}$$

i.e. $$\bar{u} = \underbrace{\overbrace{\left(\frac{\bar{u} \cdot \bar{a}}{\|\bar{a}\|^2}\right)}^{k} \bar{a}}_{\bar{w}_1} + \underbrace{\left(\bar{u} - \frac{\bar{u} \cdot \bar{a}}{\|\bar{a}\|^2} \bar{a}\right)}_{\bar{w}_2}$$

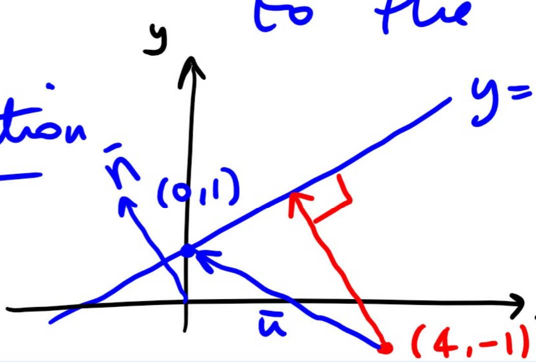
!!! $\bar{w}_2$ is just supposed to be "defined" here by $\bar{w}_2 = \bar{u} - \bar{w}_1 = \bar{u} - \frac{\bar{u} \cdot \bar{a}}{\|\bar{a}\|^2} \bar{a}$

called the <u>orthogonal projection</u> of $\bar{u}$ onto $\bar{a}$, $\text{proj}_{\bar{a}} \bar{u}$

component of $\bar{u}$ orthogonal to $\bar{a}$

Example Find the shortest distance from $(4,-1)$ to the line $y = \frac{1}{2}x + 1$.

Solution



The answer is $\|\text{proj}_{\bar{n}} \bar{u}\|$ where

$\bar{n}$ is normal to line

$\bar{u}$ joins $(4,-1)$ to $(0,1)$ on line (can take any point on the line that you know)

Find $\bar{n}$: $\quad y = (1/2)x + 1 \longrightarrow 1x - 2y + 2 = 0$

$\Rightarrow \bar{n} = (1, -2)$

Find $\bar{u}$: $\quad (0,1) - (4,-1) = (-4, \ 2)$

From Theorem: $\quad \text{distance} = \|\text{proj}_{\bar{n}} \bar{u}\| = \left\| \frac{\bar{u}\cdot\bar{n}}{\|\bar{n}\|^2}\bar{n} \right\|$

$$= \left| \frac{\bar{u}\cdot\bar{n}}{\|n\|^{\cancel{2}}} \right| \cancel{\|n\|}$$

$$= \left| \frac{(-4,2)\cdot(1,-2)}{\sqrt{1^2+(-2)^2}} \right| = \frac{|-8|}{\sqrt{5}} = \frac{8}{\sqrt{5}}$$

<u>Short cut in $\mathbb{R}^2$</u>

shortest distance from (x_0, y_0) to $ax + by + c = 0$ is

$$\frac{|ax_0 + by_0 + c|}{\sqrt{a^2+b^2}}$$

plug in

$\bar{n} = (a, b)$

$\bar{u} = (0, -\frac{c}{b}) - (x_0, y_0)$

$= (-x_0, -\frac{c}{b} - y_0)$

Where $ax + by + c = 0$ crosses y-axis

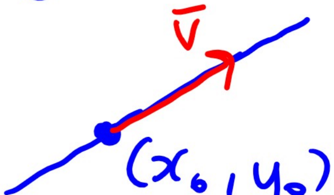
$$\frac{|\bar{u}\cdot\bar{n}|}{\|\bar{n}\|} = \frac{|-ax_0 - c - by_0|}{\sqrt{a^2+b^2}}$$

<u>In $\mathbb{R}^3$</u> Shortest distance from (x_0, y_0, z_0) to plane $ax + by + cz + d = 0$ is $\dfrac{|ax_0 + by_0 + cz_0 + d|}{\sqrt{a^2+b^2+c^2}}$.

3.4 The Geometry of Linear Systems

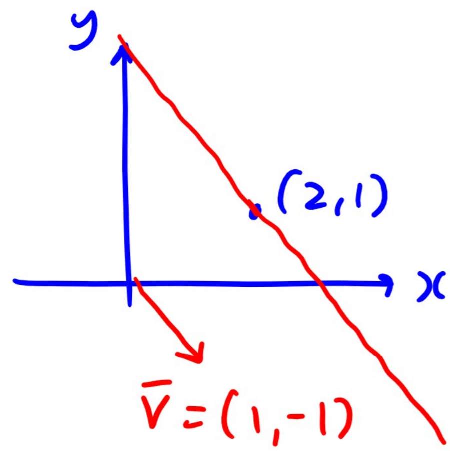
Another way to think of lines & planes.

Lines in $\mathbb{R}^2$



Every point on line is of form $(x_0, y_0) + t\bar{v}$, for some $t \in \mathbb{R}$

Example $(2,1) + t(1,-1)$



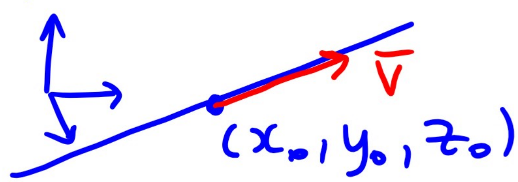
Points on line:

$$(x, y) = (2+t, 1-t)$$

"parametric form" of line

Notice $x + y = 3$ so this is the solution set to this ↑ equation

Lines in $\mathbb{R}^3$ same



Parametrically $(x, y, z) = (x_0, y_0, z_0) + t\bar{v}$, $t \in \mathbb{R}$

Example Point $(1, -1, 3)$

& Vector $(-2, 1, 1)$

traces out line with points

$$(x, y, z) = (1, -1, 3) + t(-2, 1, 1) = (1-2t, -1+t, 3+t)$$

This is parametric solution to $\begin{cases} x + 2y = -1 \\ x + 2z = 7 \end{cases}$

T.B.C. !