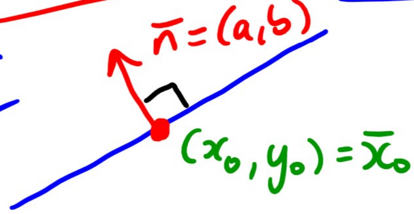
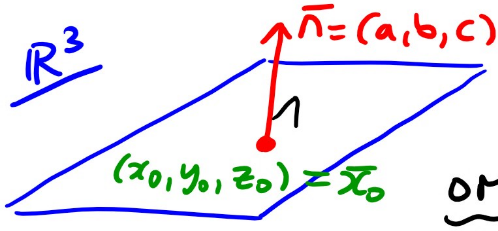
1B03 – LINEAR ALGEBRA 1 (C01) WS19 — Lecture 24

Last Time — Lines & Planes using vectors

$\mathbb{R}^2$



$\mathbb{R}^3$



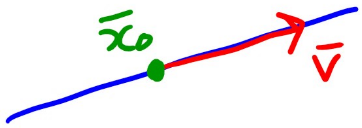
→ With a specified point and an orthogonal normal vector

$\bar{u}, \bar{v} \in \mathbb{R}^n$ orthogonal: $\bar{u} \cdot \bar{v} = 0$

Line Equation: $ax + by + c = 0$

Plane Equation: $ax + by + cz + d = 0$

In $\mathbb{R}^2$ & $\mathbb{R}^3$:

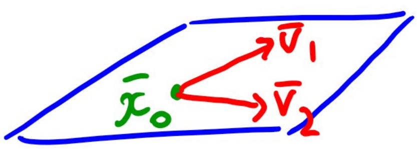


Parametric Line Equation: $\bar{x} = \bar{x}_0 + t\bar{v}$, $t \in \mathbb{R}$.

$\bar{x}$: (x, y) or (x, y, z)

$\bar{x}_0$: (x_0, y_0) or (x_0, y_0, z_0)

In $\mathbb{R}^3$:



$\bar{v}_1, \bar{v}_2$ in plane NOT colinear (but not necessarily orthogonal)

Every point on plane: $\bar{x} = (x, y, z) = \bar{x}_0 + t\bar{v}_1 + s\bar{v}_2$, $t, s \in \mathbb{R}$

$\bar{x}_0$: (x_0, y_0, z_0)

In $\mathbb{R}^n$
- the line through $\bar{x}_0 \in \mathbb{R}^n$ parallel to $\bar{v}$ is $\bar{x} = \bar{x}_0 + t\bar{v}$, $t \in \mathbb{R}$
- the plane through $\bar{x}_0 \in \mathbb{R}^n$ "parallel to" $\bar{v}_1, \bar{v}_2$ (NOT colinear) is $\bar{x} = \bar{x}_0 + t\bar{v}_1 + s\bar{v}_2$, $t, s \in \mathbb{R}$

Line equation in $\mathbb{R}^n$ is solution set to system of $n-1$ LEs

Plane " " " " " " " " $n-2$ ".

Example Find the line in $\mathbb{R}^5$ passing through the points $(1,0,-1,2,3)$ and $(5,-1,-2,0,6)$.

Solution Want 1 point $\bar{x}_0$ & 1 vector $\bar{v}$ in line:

Choose one given point as $\bar{x}_0$ say $(1,0,-1,2,3)$.

Now find $\bar{v} = (1,0,-1,2,3) - (5,-1,-2,0,6)$

$$= (-4, 1, 1, 2, -3).$$

So line is $\bar{x} = \bar{x}_0 + t\bar{v}$, $t \in \mathbb{R}$

$$= (1,0,-1,2,3) + t(-4,1,1,2,-3)$$

$= (x_1, x_2, x_3, x_4, x_5)$

$\rightarrow$
$$x_1 = 1-4t$$
$$x_2 = t$$
$$x_3 = -1+t$$
$$x_4 = 2+2t$$
$$x_5 = 3-3t.$$

The geometry of linear systems say $A\bar{x} = \bar{0}$

A is $m \times n$

m equations, each looks like:

$$\underbrace{a_{i1}x_1 + a_{i2}x_2 + \ldots + a_{in}x_n}_{\bar{a}_i \cdot \bar{x}} = 0$$

$(a_{i1}, \ldots, a_{in}) \rightarrow \bar{a}_i \cdot \bar{x} \leftarrow \bar{x} = (x_1, \ldots, x_n)$

$\uparrow$ ith row of A i.e. every solution $\bar{x}$ to $A\bar{x}=\bar{0}$ is orthogonal to every row of A.

Now suppose $A\bar{x}=\bar{b}$ is a system of LEs & we find solution $\bar{x}_0$ i.e. $A\bar{x}_0=\bar{b}$.

Then all solutions $\bar{x}$ to $A\bar{x}=\bar{b}$ look like

$$\bar{x} = \bar{x}_0 + \bar{w} \quad \text{where } A\bar{w}=\bar{0}.$$

[$A(\bar{x}_0+\bar{w}) = A\bar{x}_0 + A\bar{w} = \bar{b}+\bar{0} = \bar{b}$ & notice

if $A\bar{x}=\bar{b}$, then $\bar{x} = \bar{x}_0 + \underbrace{(\bar{x}-\bar{x}_0)}$ so this is $\bar{w}$

& $A(\bar{x}-\bar{x}_0) = A\bar{x} - A\bar{x}_0 = \bar{b}-\bar{b} = \bar{0}$]

<u>Example</u> If A is 2×3 & $A\bar{x}=\bar{0}$

2 equations; 3 variables so solutions in $\mathbb{R}^3$

From above, solution set : line

For sure $\bar{x}=\bar{0}$ is a solution i.e. line goes through origin

$\Rightarrow$ line is $t\bar{v}$, $t\in\mathbb{R}$,

where $\bar{v}$ is orthogonal to both rows of A

Now if $A\bar{x} = \bar{b}$ (same A) & $\bar{x}_0$ is <u>some</u> solution, then all solutions to $A\bar{x} = \bar{b}$ look like

Notice this is a line parallel to the solution set to $A\bar{x} = \bar{0}$, which was the $t\bar{v}$, $t \in \mathbb{R}$ part.

$$\bar{x}_0 + t\bar{v} \qquad \text{(line parallel to } t\bar{v},\ t \in \mathbb{R}\text{)}$$
$$t \in \mathbb{R}$$

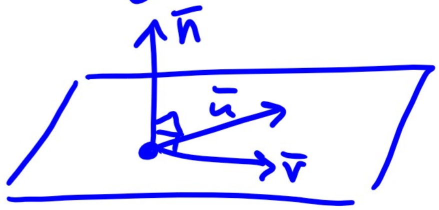
3.5 Cross Product - only in $\mathbb{R}^3$

↳ 2 vectors $\bar{u}, \bar{v}$ <u>not</u> colinear

→ want $\bar{w}$ orthogonal to both $\bar{u}, \bar{v}$

Why? - e.g. situation above (solution, 2 rows of A)

- e.g.



Find normal $\bar{n}$ to plane containing $\bar{u}, \bar{v}$

The <u>cross product</u> of $\bar{u} = (u_1, u_2, u_3)$ &
$\bar{v} = (v_1, v_2, v_3)$ is

$$\bar{u} \times \bar{v} = \begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \begin{matrix} \\ \leftarrow \bar{u} \\ \leftarrow \bar{v} \end{matrix}$$

* Not actually a determinant

(because * this grid is not a matrix!)

(pretending all the same that we can do cofactor expansion!)

$$= \bar{e}_1 \begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix} - \bar{e}_2 \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix} + \bar{e}_3 \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix}$$

$$= \left(\begin{vmatrix} u_2 & u_3 \\ v_2 & v_3 \end{vmatrix}, - \begin{vmatrix} u_1 & u_3 \\ v_1 & v_3 \end{vmatrix}, \begin{vmatrix} u_1 & u_2 \\ v_1 & v_2 \end{vmatrix} \right)$$

$$= \left(u_2 v_3 - u_3 v_2, -(u_1 v_3 - u_3 v_1), u_1 v_2 - u_2 v_1 \right)$$

<u>Check</u> $\bar{e}_1 \times \bar{e}_2 = \bar{e}_3$

$\bar{e}_2 \times \bar{e}_3 = \bar{e}_1$

$\bar{e}_3 \times \bar{e}_1 = \bar{e}_2$

Then $\bar{u} \cdot (\bar{u} \times \bar{v})$

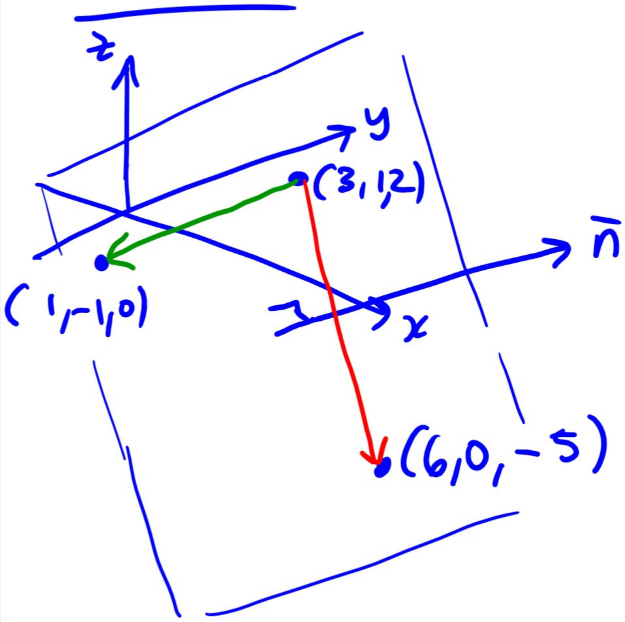
$= \bar{v} \cdot (\bar{u} \times \bar{v})$

$= 0.$

(See textbook if you need help checking!)

<u>Exercise</u> Find the plane containing $(3,1,2)$, $(6,0,-5)$ & $(1,-1,0)$.

<u>Solution</u>



Define $\bar{u} = (6,0,-5) - (3,1,2)$

$= (3,-1,-7)$

$\bar{v} = (1,-1,0) - (3,1,2)$

$= (-2,-2,-2)$

$$\bar{n} = \bar{u} \times \bar{v} = \begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ 3 & -1 & -7 \\ -2 & -2 & -2 \end{vmatrix}$$

$= (-12, 20, -8)$

Using $\bar{n}$ & one point ~~point~~, say $(3,1,2)$ can find equation $-12x + 20y - 8z + d = 0$;

plug in $(3,1,2)$ to solve for d:

$$d = 36 - 20 + 16 = 32$$

So plane: $\underbrace{-12x + 20y - 8z + 32 = 0}$

<u>Properties</u>

- $\bar{u} \times \bar{u} = \bar{0}$
- $\bar{u} \times \bar{0} = \bar{0}$

$$\begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ u_1 & u_2 & u_3 \\ u_1 & u_2 & u_3 \end{vmatrix}$$

- $\bar{v} \times \bar{u} = -(\bar{u} \times \bar{v})$ $\rightarrow$ Think right-hand rule
- $\bar{u} \times (\bar{v} + \bar{w}) = (\bar{u} \times \bar{v}) + (\bar{u} \times \bar{w})$
- $k\bar{u} \times \bar{v} = k(\bar{u} \times \bar{v})$

- (Lagrange's Identity) — multiply out to get:

$$\| \bar{u} \times \bar{v} \| = \sqrt{\|\bar{u}\|^2 \|\bar{v}\|^2 - (\bar{u} \cdot \bar{v})^2}$$

(Check!)

See Theorems 3.5.1 & 3.5.2 for more.