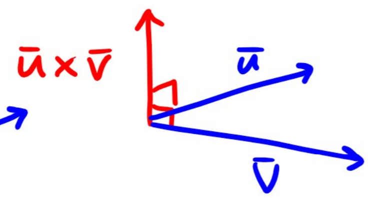
1B03 – LINEAR ALGEBRA 1 (C01) WS19 Lecture 25

Last Time: Cross Product in $\mathbb{R}^3$

$$\bar{u} \times \bar{v} = \begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} = (u_2v_3 - u_3v_2, -(u_1v_3 - u_3v_1), u_1v_2 - u_2v_1)$$

$$(\bar{u} \times \bar{v}) \cdot \bar{u} = 0 = (\bar{u} \times \bar{v}) \cdot \bar{v}$$



- $\bar{u} \times \bar{v} = -(\bar{v} \times \bar{u})$; • $\bar{u} \times \bar{0} = \bar{0}$; • $\bar{u} \times \bar{u} = \bar{0}$; etc.
- Lagrange's Identity: $\|\bar{u} \times \bar{v}\| = \sqrt{\|\bar{u}\|^2 \|\bar{v}\|^2 - (\bar{u} \cdot \bar{v})^2}$

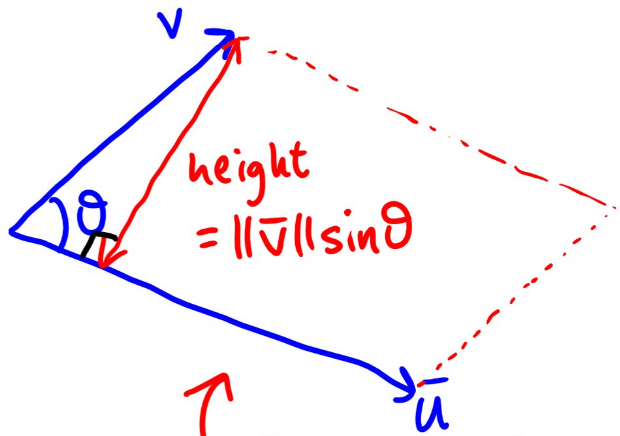
$$(\bar{u} \cdot \bar{v})^2 = \|\bar{u}\|^2 \|\bar{v}\|^2 \cos^2\theta$$

so $\|\bar{u} \times \bar{v}\| =$

$$\sqrt{\|\bar{u}\|^2 \|\bar{v}\|^2 - \|\bar{u}\|^2 \|\bar{v}\|^2 \cos^2\theta}$$

$$= \|\bar{u}\| \|\bar{v}\| \sqrt{1 - \cos^2\theta}$$

$$= \|\bar{u}\| \|\bar{v}\| \sin\theta$$

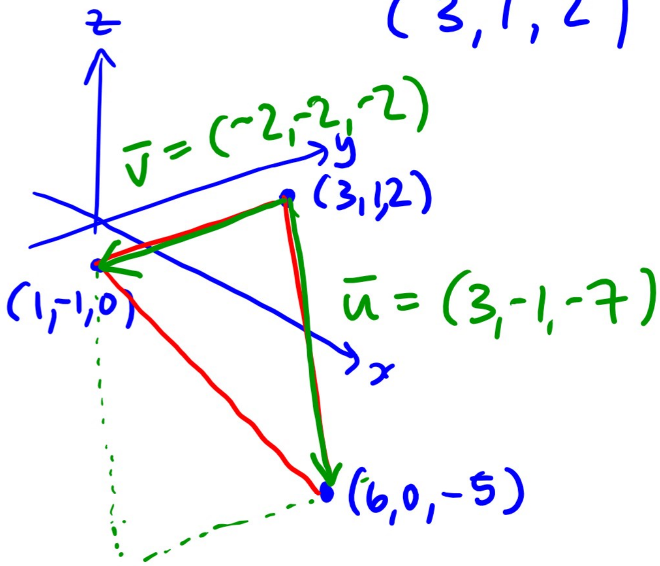


Area of parallelogram = base × height
$= \|\bar{u}\| \times \|\bar{v}\| \sin\theta$

!!!

$\frac{\text{opp.}}{\text{hyp.}} = \sin\theta$

Example Find the area of the triangle joining $(3,1,2)$, $(6,0,-5)$, $(1,-1,0)$.



Area of Δ = $\frac{1}{2}$ Area of parallelogram given by $\bar{u}$ & $\bar{v}$

$$= \frac{1}{2}\|\bar{u} \times \bar{v}\| \quad \text{last lecture}$$

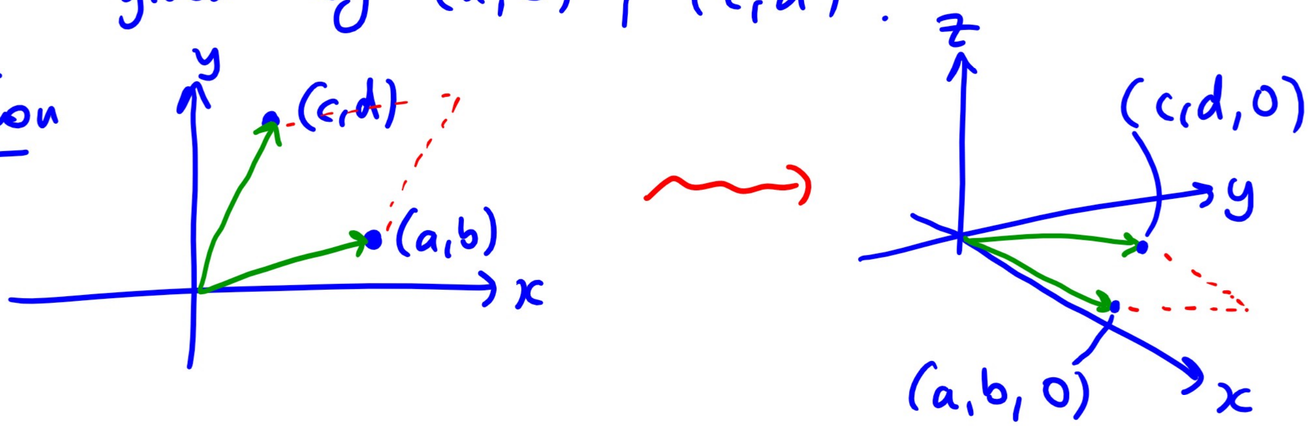
$$= \frac{1}{2}\|(-12, 20, -8)\|$$

$$= \frac{1}{2}\sqrt{(-12)^2 + 20^2 + (-8)^2} = \frac{1}{2}\sqrt{608} = \frac{1}{2}(4\sqrt{38})$$

$$\simeq \underline{\underline{12.33}}.$$

Example Find the area of the parallelogram in $\mathbb{R}^2$ given by (a,b), (c,d).

Solution



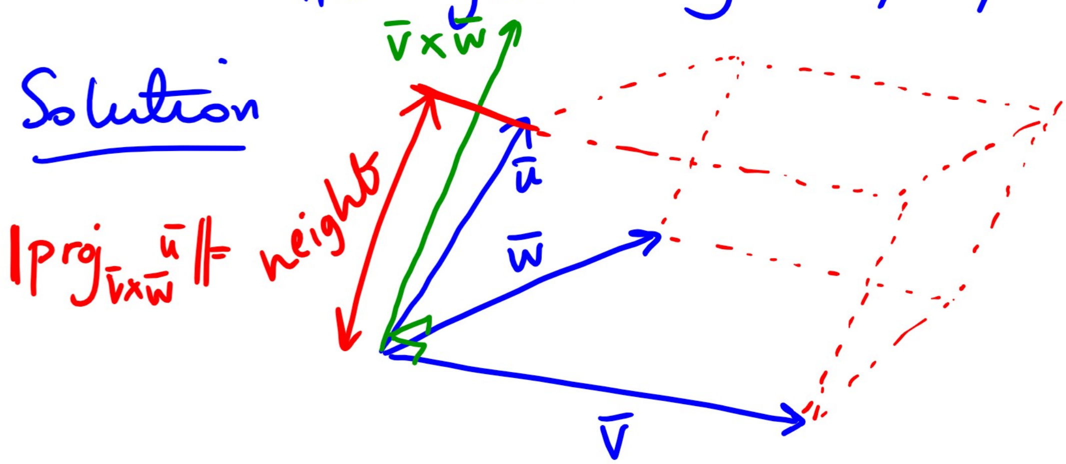
So area $= \|(a,b,0) \times (c,d,0)\|$

$$= \left\| \begin{vmatrix} \bar{e}_1 & \bar{e}_2 & \bar{e}_3 \\ a & b & 0 \\ c & d & 0 \end{vmatrix} \right\| = \|(0,0,ad-bc)\|$$

$$= |ad-bc| = \left|\det\begin{bmatrix} a & b \\ c & d \end{bmatrix}\right|$$

<u>Example</u> Find the volume of the parellelipiped in $\mathbb{R}^3$ given by $\bar{u}, \bar{v}, \bar{w}$.

<u>Solution</u>



Volume = Area of parallelogram given by $\bar{w}$ & $\bar{v}$ × height

$$= \|\bar{v}\times\bar{w}\| \cdot \|\text{proj}_{\bar{v}\times\bar{w}}\,\bar{u}\|$$

$$= \cancel{\|\bar{v}\times\bar{w}\|}\ \frac{|\bar{u}\cdot(\bar{v}\times\bar{w})|}{\cancel{\|\bar{v}\times\bar{w}\|}}$$

$\bar{u}\cdot(\bar{v}\times\bar{w})$ is the <u>scalar triple</u> <u>product</u> of $\bar{u}, \bar{v}$ & $\bar{w}$.

$$\bar{u}\cdot(\bar{v}\times\bar{w}) = \bar{u}\cdot\overbrace{\left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix}\bar{e}_1 - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix}\bar{e}_2 + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\bar{e}_3\right)}^{=\left(\begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix},\, -\begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix},\, \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix}\right)}$$

$$= \begin{vmatrix} v_2 & v_3 \\ w_2 & w_3 \end{vmatrix} u_1 - \begin{vmatrix} v_1 & v_3 \\ w_1 & w_3 \end{vmatrix} u_2 + \begin{vmatrix} v_1 & v_2 \\ w_1 & w_2 \end{vmatrix} u_3$$

$$= \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}. \quad \text{i.e. } \det\begin{bmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{bmatrix}.$$

4.1 Real Vector Spaces

your favourite n

- generalizes "the collection of all vectors in $\mathbb{R}^n$" & their interactions
- what kind of mathematical objects exhibit the 'same' ways of interacting?

  e.g. can add, subtract & negate vectors, but not multiply them — but can scale them by a real #

  & we have dot product, norm, distance etc.

  & also linear combinations of vectors, ...

Definition V – real vector space

- non-empty collection of (any kind of!!) math. objects
- with concepts of "addition" (of 2 objects) & "scalar multiplication" by $k \in \mathbb{R}$

(possibly totally unrelated to usual + & (scalar) mult. with real #s)

Satisfying 10 axioms (rules):

Notice how, without knowing which objects we're talking about & which notation we should use, the default notation is $\bar{u}$, $\bar{v}$ etc. & we even call the objects "vectors" — but they are NOT necessarily vectors i.e. line segments in $\mathbb{R}^n$!

1. V is closed under addition:

   i.e. If $\bar{u}, \bar{v} \in V$, then $\bar{u} + \bar{v} \in V$

2. $\bar{u} + \bar{v} = \bar{v} + \bar{u}$
3. $\bar{u} + (\bar{v} + \bar{w}) = (\bar{u} + \bar{v}) + \bar{w}$

   $\Big\}$ order of addition doesn't matter

4. There is an object called $\bar{0}$, the "zero vector", with $\bar{u} + \bar{0} = \bar{u}$ $(= \bar{0} + \bar{u})$ for any $\bar{u} \in V$
5. For any $\bar{u} \in V$, there's a "negative of $\bar{u}$", $-\bar{u}$, with $\bar{u} + (-\bar{u}) = \bar{0}$
6. V is "closed under scalar multiplication":

   If $\bar{u} \in V$, $k \in \mathbb{R}$, then $k\bar{u} \in V$

7. $k(\bar{u} + \bar{v}) = k\bar{u} + k\bar{v}$
8. $(k + m)\bar{u} = k\bar{u} + m\bar{u}$

   This is the usual + for real #s!!

9. $(km)\bar{u} = k(m\bar{u})$

   This is the usual multiplication for real #s

10. $1\bar{u} = \bar{u}$.