1B03 – LINEAR ALGEBRA 1 (C01) WS19 Lecture 26

Yesterday Real Vector Spaces V → non-empty collection of objects (written as $\bar{u}, \bar{v}$ etc.)

with $+$ → addition of objects

$k.$ → scalar multiplication of objects by real numbers k

satisfying AXIOMS:

A1) For all $\bar{u}, \bar{v} \in V$, $\bar{u} + \bar{v} \in V$. ← "Closure under addition"

A2) For all $\bar{u}, \bar{v} \in V$, $\bar{u} + \bar{v} = \bar{v} + \bar{u}$.

A3) For all $\bar{u}, \bar{v}, \bar{w} \in V$, $\bar{u} + (\bar{v} + \bar{w}) = (\bar{u} + \bar{v}) + \bar{w}$.

A4) There is a zero $\bar{0}$ with $\bar{u} + \bar{0} = \bar{0} + \bar{u} = \bar{u}$, for all $\bar{u} \in V$.

A5) For all $\bar{u} \in V$, there is a negative $-\bar{u}$ with $\bar{u} + (-\bar{u}) = \bar{0}$.

A6) For all $\bar{u} \in V$, $k \in \mathbb{R} \Rightarrow k.\bar{u} \in V$. ← "closure under scalar mult."

A7) For all $k \in \mathbb{R}$, $\bar{u}, \bar{v} \in V$, $k.(\bar{u} + \bar{v}) = k.\bar{u} + k.\bar{v}$.

A8) For all $k, m \in \mathbb{R}$, $\bar{u} \in V$, $(k+m).\bar{u} = k.\bar{u} + m.\bar{u}$.

A9) For all $k, m \in \mathbb{R}$, $\bar{u} \in V$, $(k.m).\bar{u} = k.(m.\bar{u})$.

A10) For all $\bar{u} \in V$, $1.\bar{u} = \bar{u}$.

Examples

① $\mathbb{R}^n$ with usual addition & scalar multiplication of vectors

fixed n

② $M_{mn}(\mathbb{R})$ with matrix addition & matrix scalar multiplication

$m \times n$ matrices with real entries

To check this, you would have to check all A1 – A10 were true in this context!

(A1) If we have A, B $m \times n$, is $A+B$ also $m \times n$? ✓

(A2) If A, B $m \times n$, is $A+B = B+A$? ✓

③ $V = \{\text{one single object}\} = \{\bar{0}\}$ Define

$$\bar{0} + \bar{0} = \bar{0} \quad \& \quad k\bar{0} = \bar{0}$$

When we define + & k. , we need to say what $\bar{u}+\bar{v}$ means and $k\bar{u}$ means for every choice of $\bar{u}, \bar{v}$ in V. Here the choice is very limited, so we gets the whole definition of + & k. by declaring what $\bar{0}+\bar{0}$ and $k\bar{0}$ are.

Then check say (A 9) $\underbrace{(km)\bar{u}}_{\substack{(km)\bar{0} \\ =\bar{0}}} = \underbrace{k(m\bar{u})}_{k(\bar{0}) = \bar{0}}$

④ $F(-\infty, \infty) = \{\text{all } \underbrace{\text{real-valued}}_{\text{range is in } \mathbb{R}} \text{ functions with domain } \underbrace{(-\infty,\infty)}_{\mathbb{R}}\}$

$(= F(\mathbb{R}))$

$\hookrightarrow f : \mathbb{R} \longrightarrow \mathbb{R}$

Notice

Objects are functions NOT the values they take.

Define $f + g : \mathbb{R} \longrightarrow \mathbb{R}$, $kf : \mathbb{R} \longrightarrow \mathbb{R}$

We define f+g and kf by declaring what their output is given input x.

$$[f+g](x) = f(x) + g(x) ,$$
$$[kf](x) = kf(x) ,$$
for all $x \in \mathbb{R}$

Check e.g. (A7) $k(\bar{u}+\bar{v}) = k\bar{u} + k\bar{v}$ for all $\bar{u}, \bar{v} \in V$

Here this looks like $\underbrace{k[f+g]} = \underbrace{kf + kg}$

To check: these 2 f^ns should agree on every $x \in \mathbb{R}$:

$$[k[f+g]](x) \underset{\substack{\uparrow \\ \text{rule for scalar mult.}}}{=} k([f+g](x)) \underset{\substack{\uparrow \\ \text{rule for +}}}{=} k(f(x) + g(x)) \underset{\substack{\uparrow \\ \text{property of } \mathbb{R}\text{, real \#s}}}{=} kf(x) + kg(x)$$

$= [kf](x) + [kg](x) = [kf + kg](x)$

rule for scalar mult.   rule for +   ✓

(5) P_d = polynomials of degree at most d

$$\{a_0 + a_1 x + a_2 x^2 + \dots + a_d x^d \ , \ a_i \in \mathbb{R}\}$$

same + & scalar mult. as in (4) $F(-\infty, \infty)$

(6) p. 188 Ex. 8: $\mathbb{R}$ with $u + v = uv$ ← usual mult. of reals

$ku = u^k$ ← usual sense of exponents

e.g. $2+3 = 6$

$5(2) = 2^5 = 32$

i.e. sometimes recognizable objects can have strange + & scalar mult. rules & still get a vector space (see textbook)

But also not:

→ Whatever you decide the + operation is, it must satisfy (A1)-(A5), (A7), and whatever you decide the scalar mult. is, it must satisfy (A6)-(A10).

<u>Non-Examples</u> (A) p. 188 Ex. 7 $\mathbb{R}^2$,

$\bar{u} + \bar{v}$ usual + $\left[(u_1, u_2) + (v_1, v_2) = (u_1 + v_1, u_2 + v_2) \right]$

define $k\bar{u} = (ku_1, 0)$

Now (A1)-(A9) true <u>but</u> (A10) not: ← Clue is that second entry always turns to 0 with scalar mult.

Need a counterexample: an explicit example of something going wrong

e.g. $1(2,5) = (2,0) \neq (2,5)$.

↑ choose any example which does not have 0 here.

(B) $\mathbb{R}^2$ usual scalar mult.

$$\bar{u} + \bar{v} = (|u_1| + |v_1|, |u_2| + |v_2|)$$

always has non-negative entries

Here (A4) fails: no zero vector.

e.g. if we had $\bar{0} = (v_1, v_2)$

then $(-2,3) + \bar{0} = (\underbrace{2+|v_1|}_{\geq 2}, \underbrace{3+|v_2|}_{\geq 3})$

$\neq (-2,3)$

pick as your counterexample anything with at least one negative entry.

Useful Facts about Vector Spaces

V vector space, $k \in \mathbb{R}$, $\bar{u} \in V$:

(a) $0\bar{u} = \bar{0}$

(b) $k\bar{0} = \bar{0}$

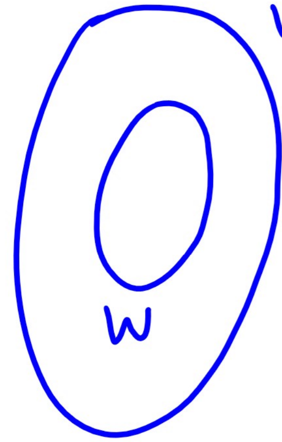
(c) $(-1)\bar{u} = -\bar{u}$

(d) If $k\bar{u} = \bar{0}$, then either $k=0$ or $\bar{u} = \bar{0}$

For an example proof of this, see page 7 of this document.

4.2 Subspaces

V vector space



W subset of v.s. V

When is W (with same + & scalar mult. ops) itself a vector space?

operations

↳ if yes, call it a **Subspace** of V

Since members of W are also in V,

we get (A2), (A3), (A7), (A8), (A9), (A10) for free

(true for objects in V)

closure under + for W

there is a $\bar{0}$ in W

every vector in W has a negative in W

closure under scalar multiplication for W

What about (A1), (A4), (A5), (A6) ?

if (A6) true → closure under scalar mult. for W

then Facts above (about V)

scalar multiples of elements of W

tell us $0\bar{u} = \bar{0}$ and $(-1)\bar{u} = -\bar{u}$ must be in W for any $\bar{u} \in W$.

i.e. by the Fact

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So (A4) & (A5) true.

(They are also in V so the facts above apply.)

So it's enough to check W not empty & (A1) & (A6) true for W.

Test for Subspaces If V is a vector space and W is a subset of V with the same operations of addition & scalar multiplication on objects as V has, then W is a subspace of V if

1. W is not empty;
2. W is closed under addition (i.e. (A1) true about W);
3. W is closed under scalar multiplication (i.e. (A6) true about W).

So if you have a collection of objects W with an addition operation & a scalar multiplication operation, but you can see that W is part of a bigger collection V which is a vector space when you use the same addition operation & scalar multiplication operation, then you can change the question "Is W a vector space?" into "Is W non-empty? Is W closed under addition & scalar multiplication?" — instead of having to check 10+ things, you only have to check 3 !

Example proofs of Facts using Axioms

V vector space

– see textbook for (a), (c) (Theorem 4.1.1 p. 189).

(b) $k\bar{0} = \bar{0}$, for $k \in \mathbb{R}$.

$k\bar{0} = k(\bar{0} + \bar{0})$ (A4) – $\bar{0} + \bar{u} = \bar{u}$ for any $\bar{u} \in V$, so in particular $\bar{u} = \bar{0}$.

$k\bar{0} = k\bar{0} + k\bar{0}$ (A7)

So $\underbrace{k\bar{0} + (-(k\bar{0}))} = \underbrace{(k\bar{0} + k\bar{0}) + (-(k\bar{0}))}$ (add $-(k\bar{0})$ to both sides)

$= \bar{0}$ (A5)

$= k\bar{0} + (k\bar{0} + (-(k\bar{0})))$ (A3)

$= k\bar{0} + \bar{0}$ (A5)

$= k\bar{0}$ (A4)

i.e. $\bar{0} = k\bar{0}$ as required.

(d) If $k\bar{u} = \bar{0}$, then either $k = 0$ or $\bar{u} = \bar{0}$.

Suppose $k\bar{u} = \bar{0}$. If $k \neq 0$, then we can consider

$$\left(\frac{1}{k}\right)k\bar{u} = \left(\frac{1}{k}\right)\bar{0}$$

$$= \bar{0} \quad \text{by (b)}$$

But of course $\left(\frac{1}{k}\right)k\bar{u} = \left(\frac{1}{k}k\right)\bar{u}$ (A9)

$= 1\bar{u}$

$= \bar{u}$ (A10)

i.e. $\bar{u} = \bar{0}$.

So either $k = 0$ or $\bar{u} = \bar{0}$ (& perhaps both).