1B03 – LINEAR ALGEBRA 1 (C01) WS19 Lecture 27

Last Time — Subspaces of Vector Spaces

V : vector space with addition & scalar multiplication ops.

W : subset of V with <u>same</u> addition & scalar mult. ops.

Test for Subspaces If W, V are as above, then W is a <u>subspace</u> of V (i.e. itself a vector space with same ops. as V)

if ① W is not empty;
② W is closed under addition;
③ W is closed under scalar multiplication.

Examples (SS#) & Non-Examples (NSS#)

Any vector space V always has 2 "trivial" subspaces:

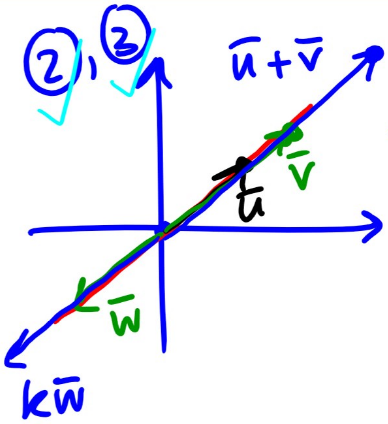
SS1 $W = V$

V satisfies all 10 axioms & is non-empty so definitely ①, ②, ③!

SS2 $W = \{\bar{0}\}$

Check: ① $\bar{0} \in \{\bar{0}\}$ ✓
② $\bar{0} + \bar{0} = \bar{0} \in W$ ✓
③ $k\bar{0} = \bar{0} \in W$ ✓

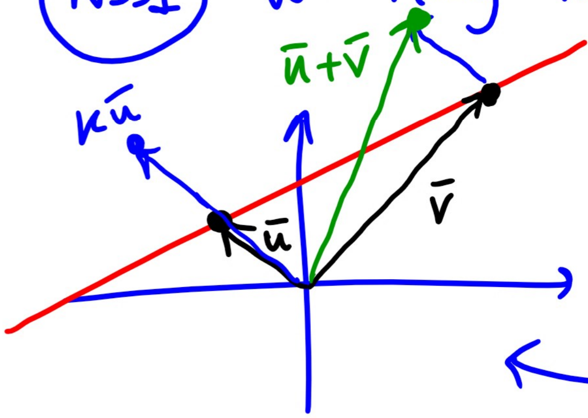
SS3 $W =$ A line (respectively a line or plane) in $\mathbb{R}^2 = V$ (resp. in $\mathbb{R}^3 = V$) <u>through</u>



the origin

① $\bar{0}$ on the line ✓

NSS1 W = Any line in $\mathbb{R}^2$ NOT through the origin
" V

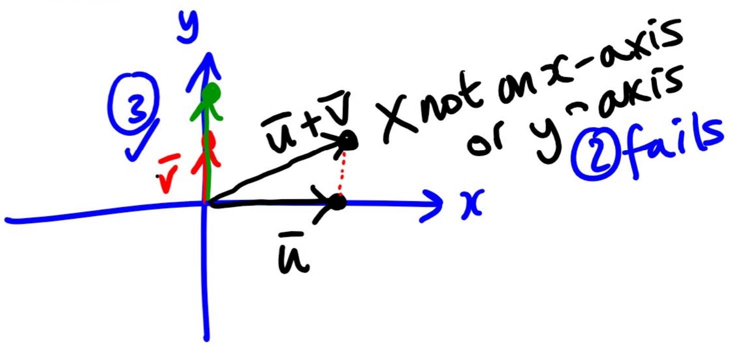


(think in terms of pairs of #s (i.e. points) on the line.)

① ✓

② & ③ both fail

NSS 2 W = All points on x-axis or y-axis in $\mathbb{R}^2 = V$



① ✓ e.g. $\bar{0}$ (it's actually on both!)

③ ✓

② X

NSS 3 W = The set of vectors $(a,b,c) \in \overbrace{\mathbb{R}^3}^{V}$ with $ab=c$

① $\bar{0}$ or $(2, 3, 6)$ or ... ✓

③ Take (a,b,c) with $ab=c$.
Look at $k(a,b,c) = (ka, kb, kc)$.

Now have to check $(ka)(kb) \stackrel{?}{=} kc$ (to check if is in W)

$$\Rightarrow k^2 ab \stackrel{?}{=} kc$$

$$\Rightarrow k^2 c \stackrel{?}{=} kc \quad (\text{use } \underline{ab = c})$$

Now cook up example with $k \neq 0, 1$
$(a, b, c) \neq \bar{0}$

to show ③ fails e.g. $5(2, 3, 6)$
$= (10, 15, 30)$
$10 \times 15 \neq 30.$

Similar idea in case ②. (also fails).
↳ See end of doc.

554 $W =$ diagonal matrices in $M_{22}(\mathbb{R})$.
(same for any choice of n)

✓ ① $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ or ... lots!

② If $A = \begin{bmatrix} a_1 & 0 \\ 0 & a_2 \end{bmatrix}$, $B = \begin{bmatrix} b_1 & 0 \\ 0 & b_2 \end{bmatrix}$

then $A + B = \begin{bmatrix} a_1 + b_1 & 0 \\ 0 & a_2 + b_2 \end{bmatrix}$ diagonal ✓

③ If $k \in \mathbb{R}$, $kA = \begin{bmatrix} ka_1 & 0 \\ 0 & ka_2 \end{bmatrix}$ diagonal ✓

Same method works for $W =$ UT matrices, $W =$ LT matrices
but NOT $W =$ triangular!!

e.g. $\begin{bmatrix} 1 & 0 \\ 2 & 3 \end{bmatrix} + \begin{bmatrix} 3 & -1 \\ 0 & 5 \end{bmatrix} = \begin{bmatrix} 4 & -1 \\ 2 & 8 \end{bmatrix}$ ② fails.

LT UT NOT triangular

triangular + triangular = not triangular

SS5 $W = P$ = set of all polynomials

↳ subspace of $F(-\infty, \infty)$ ← functions $f: \mathbb{R} \to \mathbb{R}$

① There are polys! e.g. $p(x) = 0$, $p(x) = x^2, \ldots$ ② sum of 2 polys is a poly. ③ scale a poly. & you still have a poly.

SS6 $W = P_d$ = polys of degree $\leq d$.

↳ subspace of P & (hence) subspace of $F(-\infty, \infty)$

① $p(x) = 0$ has degree $\leq d$. So does $p(x) = x^d$. Make sure your choice has degree $\leq d$.

② $(a_0 + a_1 x + \ldots + a_d x^d) + (b_0 + b_1 x + \ldots + b_d x^d) = (a_0 + b_0) + \ldots + (a_d + b_d) x^d$ → this is a poly. of deg. $\leq d$ ✓

③ $k(a_0 + a_1 x + \ldots + a_d x^d) = (ka_0) + (ka_1) x + \ldots + (ka_d) x^d$ → This is a poly. of degree $\leq d$ ✓

Nested spaces of functions: see Textbook pp. 194–195.

3 really important types of subspaces

<u>Definition</u> If V is a vector space and $S = \{\bar{w}_1, \bar{w}_2, \ldots, \bar{w}_k\}$ is a collection of vectors in V, then

the <u>span of S</u> = { all linear combinations of $\bar{w}_1, \dots, \bar{w}_k$ }

is a subspace of V.

vectors that look like $t_1\bar{w}_1 + \dots + t_k\bar{w}_k$ for some choice of scalars $t_1, \dots, t_k$.

sometimes called the "subspace of V <u>generated</u> by $\bar{w}_1, \dots \bar{w}_k$." AKA "generated by S."

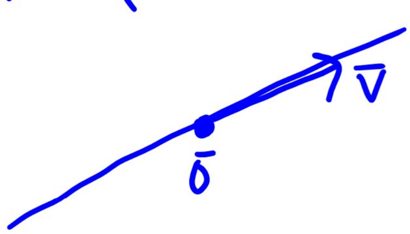
It is the "smallest" subspace of V containing all of $\bar{w}_i$ in S.

SS7 In $\mathbb{R}^n$, $\text{span}(\{\bar{e}_1, \dots, \bar{e}_n\}) = \mathbb{R}^n$

$\to$ every $\bar{v} = (v_1, \dots, v_n)$ in $\mathbb{R}^n$

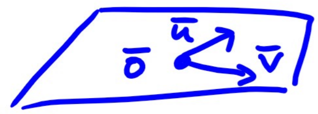
can be written as $v_1\bar{e}_1 + v_2\bar{e}_2 + \dots + v_n\bar{e}_n$

SS8 If $\bar{v} \in \mathbb{R}^n$, then $\text{span}(\{\bar{v}\})$ is line through origin, direction $\bar{v}$



It has parametric equation $\bar{x} = t\bar{v}$.

If $\bar{u}, \bar{v} \in \mathbb{R}^n$, then $\text{span}(\{\bar{u}, \bar{v}\})$ is plane through origin parallel to $\bar{u}, \bar{v}$ (as long as $\bar{u}, \bar{v}$ NOT colinear)



(S59) All polynomials of degree ≤ 3 (P_3)

look like $\underbrace{a_0}_{a_0(1)} + a_1 \underbrace{x} + a_2 \underbrace{x^2} + a_3 \underbrace{x^3}$

$\rightarrow$ linear comb. of $\{1, x, x^2, x^3\}$.

Each of these is a "vector" (i.e. object) in P_3.

So $\text{span}(\{1, x, x^2, x^3\}) = P_3$.

<u>Question</u> Does $\{1, x - x^2, x^3, 1 + x^2\}$ span P_3 too?

i.e. can we write every poly. of degree ≤ 3 as a linear comb. of these "vectors"?

T.B.C.

$V = \mathbb{R}^3$, $W = \{(a, b, c) \in \mathbb{R}^3 \mid ab = c\}$ is NOT closed under addition : take (a_1, b_1, c_1), (a_2, b_2, c_2) in W.

$(a_1, b_1, c_1) + (a_2, b_2, c_2) = (\underbrace{a_1 + a_2}, \underbrace{b_1 + b_2}, c_1 + c_2)$

$(a_1 + a_2)(b_1 + b_2) = \underbrace{a_1 b_1} + \underbrace{a_2 b_2} + a_1 b_2 + a_2 b_1$

$= c_1 + c_2 + a_1 b_2 + a_2 b_1$

This is not equal to $c_1 + c_2$ if $a_1 b_2 + a_2 b_1 \neq 0$. So, to cook up a counterexample to 2, find two vectors satisfying

$a_1 b_2 + a_2 b_1 \neq 0$ and $a_1 b_1 = c_1$ and $a_2 b_2 = c_2$.

e.g. choose $a_1 = 1$, $a_2 = 1$, $b_1 = 1$ & then as long as $b_2 \neq -1$, we're OK e.g. $b_2 = 1$.

Then $a_1 b_1 = 1$ & $a_2 b_2 = 1$ so $c_1 = 1$, $c_2 = 1$ i.e

$$(\underbrace{1, 1}_{1 \times 1 = 1}, \underbrace{1}) + (\underbrace{1, 1}_{1 \times 1 = 1}, \underbrace{1}) = (\underbrace{2, 2}_{2 \times 2 \neq 2}, \underbrace{2}).$$

So 2 fails. W is not closed under addition.