1B03 – LINEAR ALGEBRA 1 (C01) WS19 Lecture 28

Last Time The span of $S = \{\bar{v}_1, \ldots, \bar{v}_r\}$ ← a bunch of vectors in vector space is

$$\text{Span}(S) = \{\underbrace{k_1\bar{v}_1 + \ldots + k_r\bar{v}_r}_{\text{all linear combinations of } \bar{v}_1, \ldots, \bar{v}_r.} \text{ for all choices of scalars } k_1, \ldots, k_r\}$$

→ It is a subspace of V (though in general S won't be).

e.g. • line in $\mathbb{R}^n$ = $\text{Span}(\{\bar{v}\})$ • plane in $\mathbb{R}^n$ = $\text{Span}(\{\bar{u}, \bar{v}\})$ ($\bar{u}, \bar{v}$ NOT colinear)

• $P_d = \text{Span}(\{1, x, \ldots, x^d\})$

Question Does $S = \{1, x - x^2, x^3, 1 + x^3\}$ span P_3 too?

The question really is: can we write every poly in P_3

i.e. every $a_0 + a_1x + a_2x^2 + a_3x^3$ as

a linear comb. of "vectors" in S i.e. as

$$b_0(1) + b_1(x - x^2) + b_2x^3 + b_3(1 + x^3) \; ?$$

Set equal & want to find b_i in terms of a_j?

$$a_0 + a_1x + a_2x^2 + a_3x^3 = (b_0 + b_3) + b_1x - b_1x^2 + (b_2 + b_3)x^3.$$

Want $a_0 = b_0 + b_3$ $\quad a_2 = -b_1$

$a_1 = b_1$ $\quad a_3 = b_2 + b_3$

So we would need $b_1 = \underbrace{a_1 = -a_2}$. This is not necessarily the case for every polynomial in P_3

Example of failure $p(x) = x^2$ $\quad a_1 = 0,\ a_2 = 1$

So no possible way to choose b_i in general, so $\mathrm{span}(S) \neq P_3$.

See end of document for the question posed at the end of the last lecture.

Example #2 of a special type of subspace

Take an $m \times n$ matrix A.

Def^n The set of solutions $\bar{x}$ to $A\bar{x} = \bar{0}$ is a subspace of $\mathbb{R}^n$ called the <u>null space of A</u> or <u>kernel of A</u> (or <u>solution space of $A\bar{x} = \bar{0}$</u>)

Test (1) $\bar{x} = \bar{0}$ is a solution ✓

(2) If $\bar{x}_1, \bar{x}_2$ are solutions, then

$$A(\bar{x}_1 + \bar{x}_2) = A\bar{x}_1 + A\bar{x}_2 = \bar{0} + \bar{0} = \bar{0} \checkmark$$

(3) If $\bar{x}$ is a solution, k is a scalar, then ... $A(k\bar{x}) = k(A\bar{x}) = k\bar{0} = \bar{0}$ ✓

Example $A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & -2 & 2 \end{bmatrix}$ has solutions to $A\bar{x} = \bar{0}$ given

by RREF of augmented matrix

$$\hookrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 2 & 0 \\ 0 & 1 & 1 & 0 \end{array}\right] \begin{array}{l} \longrightarrow x = -2t \\ \longrightarrow y = -t \end{array}$$

...

$z = t$ solutions are $t\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$

So nullspace of A is $\text{Span}\left\{\begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}\right\}$.

Example #3 of a special type of subspace

A $n \times n$, λ eigenvalue of A

Def^n The λ-eigenspace of A is

$$E_\lambda = \{\bar{x} \text{ with } A\bar{x} = \lambda\bar{x}\},$$ a subspace of $\mathbb{R}^n$.

Test ① $\bar{0} \in E_\lambda$ ✓

② If $\bar{x}_1, \bar{x}_2 \in E_\lambda$, then $A(\bar{x}_1 + \bar{x}_2) = A\bar{x}_1 + A\bar{x}_2 = \lambda\bar{x}_1 + \lambda\bar{x}_2$
$= \lambda(\bar{x}_1 + \bar{x}_2)$ ✓

③ If $A\bar{x} = \lambda\bar{x}$ (i.e. $\bar{x} \in E_\lambda$) and k is a scalar

then $A(k\bar{x}) = k(A\bar{x}) = k(\lambda\bar{x}) = \lambda(k\bar{x})$ ✓

Example In Lecture 15 we found $\lambda = 5$ is an eigenvalue of $A = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 4 & 3 \end{bmatrix}$.

We said "a basis for eigenspace of $\lambda = 5$ is

$$\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix} \right\}."$$

This means (amongst other things) that

$$E_5 = \text{Span}\left(\left\{ \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1/2 \\ 1 \end{pmatrix} \right\} \right)$$

Notice Eigenvectors are non-zero solutions to $(A - \lambda I)\bar{x} = \bar{0}$

So really eigenspaces are special null spaces.

Question: If W is a subspace of vector space V, is there always a bunch of vectors S with $\text{span}(S) = W$?

(Well, could take $S = W$, but better to ask: S finite? How small? (small = good)

An observation about spans: (Thm 4.2.6)

If $S = \{\bar{v}_1, \dots, \bar{v}_r\}$ and $T = \{\bar{w}_1, \dots, \bar{w}_m\}$ are sets of vectors in V then:

$\mathrm{span}(S) = \mathrm{span}(T)$ exactly when every $\bar{v}_i \in \mathrm{span}(T)$ & every $\bar{w}_j \in \mathrm{span}(S)$

Example

$$\mathrm{Span}\left(\left\{\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}0\\-1\\-1\end{pmatrix}\right\}\right.$$

$$= \mathrm{Span}\left(\left\{\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}\right\}\right)$$

$$= \mathrm{Span}\left(\left\{\begin{pmatrix}0\\1\\1\end{pmatrix}, \begin{pmatrix}1\\2\\3\end{pmatrix}, \begin{pmatrix}1\\3\\4\end{pmatrix}\right\}\right..$$

UP TO HERE FOR TEST #2

4.3 Linear Independence "no redundancy".

Def^n $S = \{\bar{v}_1, \dots, \bar{v}_r\}$ is linearly independent if the only choice of scalars $k_1, \dots, k_r$ making $k_1\bar{v}_1 + k_2\bar{v}_2 + \dots + k_r\bar{v}_r = \bar{0}$ is $0 = k_1 = k_2 = \dots = k_r$.

(S otherwise linearly dependent.)

$\hookrightarrow \underbrace{S \text{ lin. dependent}}$: at least one vector in S can be written as a linear combination of the others.

This says "S linearly dependent"

Suppose we have $k_1, \dots, k_r$ not all zero e.g. $k_2 \neq 0$ with $k_1\bar{v}_1 + \dots + k_r\bar{v}_r = \bar{0}$ then e.g.

say $\bar{v}_2 = -\frac{1}{k_2}(\underbrace{k_1\bar{v}_1 + \dots + k_r\bar{v}_r}_{\text{This should say } k_1\bar{v}_1 + k_3\bar{v}_3 + \dots + k_r\bar{v}_r})$ we moved $k_2\bar{v}_2$ to the other side

Example $\{\bar{e}_1, \dots, \bar{e}_n\}$ is linearly independent in $\mathbb{R}^n$

$$\bar{0} = k_1\bar{e}_1 + \dots + k_n\bar{e}_n = (k_1, \dots, k_n) \Rightarrow k_i = 0 \text{ for all } i \checkmark$$

Example Is $S = \{(1,0,0,1,1), (3,1,-1,0,2), (0,-2,1,3,0)\}$ linearly dependent?

Solution This question means: can we write $\bar{0}$ as a linear combination of the vectors in S?

i.e. is there a non-zero solution to $\llcorner$: the system of LEs whose augmented matrix is

$$\left[\begin{array}{ccc|c} 1 & 3 & 0 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 3 & 0 \\ 1 & 2 & 0 & 0 \end{array}\right] ?$$

This comes from: $k_1(1,0,0,1,1) + k_2(3,1,-1,0,2) + k_3(0,-2,1,3,0) = \bar{0}$.

Row Reduce: We get $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$. So the only solution is $k_1 = k_2 = k_3 = 0$

This means that the answer is <u>no</u> we can't find a way to write $\bar{0}$ as a linear combination of the vectors in S, and so in fact S is linearly <u>in</u>dependent, not linearly dependent.

Solution to the problem posed last time:

Does $\{1, x-x^2, x^3, 1+x^2\}$ span P_3?

i.e. can we write every poly. $a_0 + a_1 x + a_2 x^2 + a_3 x^3$

as $b_0(1) + b_1(x-x^2) + b_2 x^3 + b_3(1+x^2)$?

i.e. can we find b_0, b_1, b_2, b_3 in terms of a_0, a_1, a_2, a_3 so that

$$a_0 + a_1 x + a_2 x^2 + a_3 x^3 =$$
$$\underbrace{(b_0+b_3)}_{=a_0} + \underbrace{b_1}_{=a_1} x + \underbrace{(b_3 - b_1)}_{=a_2} x^2 + \underbrace{b_2}_{a_3} x^3 \ ?$$

Question is: can we solve this linear system?

- $b_1 = a_1$ • $b_3 - b_1 = a_2 \rightsquigarrow b_3 = a_2 + b_1 = a_2 + a_1$
- $b_2 = a_3$ • $b_0 + b_3 = a_0 \rightsquigarrow b_0 = a_0 - b_3 = a_0 - (a_2 + a_1)$

So yes, we can & so $\{1, x-x^2, x^3, 1+x^2\}$ spans P_3.