1B03 – LINEAR ALGEBRA 1 (C01) WS19 Lecture 29

Yesterday Linear Independence

$S = \{\bar{v}_1, \dots, \bar{v}_r\}$ is linearly independent if the only choice of scalars $k_1, \dots, k_r$ with $k_1\bar{v}_1 + \dots + k_r\bar{v}_r = \bar{0}$, is $k_i = 0$ for all i.

↳ This says "There's no way to write any vector $\bar{v}_i$ from S as a "non-trivial" linear combination of (some of) the other vectors in S."

To check: set $k_1\bar{v}_1 + \dots + k_r\bar{v}_r = \bar{0}$ and solve for $k_1, \dots, k_r$.

Example $\{1, x, x^2, \dots, x^d\}$ is linearly independent in P_d.

Set $k_0 + k_1 x + k_2 x^2 + \dots + k_d x^d = 0$ (for all x) ← $p(x) = 0$ for all x

We get to compare coefficients : $k_0 = k_1 = \dots = k_d = 0$.

(Think about this!)

Example Is $S = \{1+x^3, x-x^2, x^2-1, x+x^3\}$ linearly independent in P_3?

Solution Set $k_1(1+x^3) + k_2(x-x^2) + k_3(x^2-1) + k_4(x+x^3) = 0$

(**Notice** $(1+x^3) + (x-x^2) + (x^2-1) - (x+x^3) = 0$. So not lin. indep.)

$$(k_1 - k_3) + (k_2 + k_4)x + (k_3 - k_2)x^2 + (k_1 + k_4)x^3 = 0.$$

4 equations
$$\begin{aligned} k_1 \quad - k_3 \quad &= 0 \\ k_2 \quad + k_4 &= 0 \\ -k_2 + k_3 \quad &= 0 \\ k_1 \quad + k_4 &= 0 \end{aligned}$$

So now reduce
$$\left[\begin{array}{cccc|c} 1 & 0 & -1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \end{array}\right] \dots \to$$

We know from above that we will find at least one non-zero solution, hence ∞-many solns.

<u>Example</u> Show that $\{\sin(x), \cos(x)\}$ is linearly independent in $F(-\infty, \infty)$.

<u>Solution</u> Let k_1, k_2 be such that
$$k_1 \sin(x) + k_2 \cos(x) = 0.$$

Here: be crafty! e.g. set $x = 0$:
$$k_1 \underbrace{\sin(0)}_{=0} + k_2 \underbrace{\cos(0)}_{=1} = 0 \quad \Rightarrow k_2 = 0$$

e.g. $x = \frac{\pi}{2}$: $k_1 \underbrace{\sin\left(\frac{\pi}{2}\right)}_{=1} + k_2 \underbrace{\cos\left(\frac{\pi}{2}\right)}_{=0} = 0 \Rightarrow k_1 = 0$

In general: If $\{f_1(x), \dots, f_n(x)\}$ is a linearly dependent set AND f_i are all

$n-1$ times differentiable, then:

- there are $k_1, \dots, k_n$ <u>not</u> all zero with

$$k_1 f_1(x) + \dots + k_n f_n(x) = 0 \text{ for all } x$$

- differentiating: $k_1 f_1'(x) + \dots + k_n f_n'(x) = 0$ for all x

$\vdots$

- keep going: $k_1 f_1^{(n-1)}(x) + \dots + k_n f_n^{(n-1)}(x) = 0$ for all x

n equations in n unknowns $k_1, \dots, k_n$

To say "$k_1, \dots, k_n$ not all zero" says there's a non-trivial solution to:

$$\boxed{\begin{bmatrix} f_1(x) & \cdots & f_n(x) \\ f_1'(x) & \cdots & f_n'(x) \\ \vdots & \ddots & \vdots \\ f_1^{(n-1)}(x) & \cdots & f_n^{(n-1)}(x) \end{bmatrix}} \begin{bmatrix} k_1 \\ \vdots \\ k_n \end{bmatrix} = \begin{bmatrix} 0 \\ \vdots \\ 0 \end{bmatrix}$$

square

The determinant of this matrix is ~~W(x)~~ $W(x)$ called the <u>Wronskian of $f_1, \dots, f_n$</u>, $W(x)$

i.e. $\det \cancel{W(x)} = 0$ for every x.

<u>Fact</u> If $f_1(x), \dots, f_n(x)$ are $n-1$ times differentiable, then $\{f_1(x), \dots, f_n(x)\}$ is linearly <u>in</u>dependent

if $W(x) \neq 0$ for some x.

<u>Example</u> $\{1, e^x, \sin x\}$ is linearly independent.

$$W(x) = \det \begin{bmatrix} 1 & e^x & \sin x \\ 0 & e^x & \cos x \\ 0 & e^x & -\sin x \end{bmatrix} = 1 \begin{vmatrix} e^x & \cos x \\ e^x & -\sin x \end{vmatrix}$$

$$\det \begin{bmatrix} 1 & e^x & \sin x \\ (1)' & (e^x)' & (\sin x)' \\ (1)'' & (e^x)'' & (\sin x)'' \end{bmatrix}$$

$$= e^x(-\sin x) - e^x \cos x$$

$$= -e^x(\cos x + \sin x)$$

Sometimes $\neq 0$ e.g. when $x = 0$: $W(0) = -1$.

<u>Facts</u> (1) If a finite $S = \{\bar{v}_1, \dots, \bar{v}_r\}$ contains $\bar{0}$ then S linearly dependent

(Question:
[If $k_1\bar{v}_1 + k_2\bar{v}_2 + \dots + k_n\bar{0} + \dots + k_r\bar{v}_r = \bar{0}$
can we find $k_1, \dots, k_n$ not all zero to make this true?
Answer: yes: choose $k_i = 0$ for $i \neq n$
& k_n = anything $\neq 0$]

(2) If $S = \{\bar{v}\}$, then S is linearly independent (unless $\bar{v} = \bar{0}$) ($k\bar{v} = 0 \Rightarrow k = 0$ unless $\bar{v} = \bar{0}$)

(3) If $S = \{\bar{u}, \bar{v}\}$, then S is linearly independent as long as $\bar{u}, \bar{v}$ not colinear

(4) If $S = \{\bar{v}_1, \dots, \bar{v}_r\} \subseteq \mathbb{R}^n$, and $r > n$,

then S is linearly dependent. (r unknowns > n equations, homogeneous system ⇒ non-trivial solns)

4.4 Coordinates and Bases ← plural of basis not base

If we have a vector in $\mathbb{R}^3$ e.g. $(1,-3,2)$ coordinates are given relative to x, y, z-axes

Shift perspective:

$$(1,-3,2) = 1\bar{e}_1 - 3\bar{e}_2 + 2\bar{e}_3$$

the coordinates are really the coefficients of $\bar{e}_1, \bar{e}_2, \bar{e}_3$ <u>in order</u>.