1B03 – LINEAR ALGEBRA 1 (C01) WS19 Lecture 3

Yesterday

left-most non-zero entry $[0 \cdots 0 \bullet \cdots]$

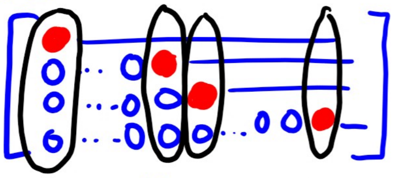
REF

RREF

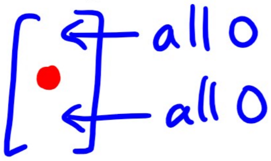
(1) The leading entry in every non-zero row is 1. $\bullet = 1$

(2) Every zero row: $[0\ 0 \ldots 0]$ is at the bottom.

(3) Every leading entry is to the right of the leading entries in the rows above (i.e. they stagger right & down:



(4) If a column has a leading entry, all its other entries are 0:



(R)REF = (Reduced) Row Echelon Form.

Definitions A column with a leading entry is a pivot column

Variables corresponding to non-pivot columns are called free variables.

Examples

Leading entries marked by /

$$\begin{bmatrix} \cancel{1} & 2 & 1 & -5 & 0 \\ 0 & \boxed{\cancel{2}}\overset{\text{Not}}{1} & 0 & 3 & 1 \\ 0 & 0 & \cancel{1} & 0 & 6 \end{bmatrix}$$

✗REF (fails (1))

$$\begin{bmatrix} \cancel{1} & \overset{\text{Not } 0}{\boxed{2}} & \overset{\text{Not } 0}{\boxed{1}} & -5 & 0 \\ 0 & \cancel{1} & 0 & 3 & 1 \\ 0 & 0 & \cancel{1} & 0 & 6 \end{bmatrix}$$

✓REF ✗RREF (fails (4))

$$\left[\begin{array}{ccccc} 0 & 0 & \boxed{1} & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \\ \boxed{1} & 1 & 0 & 3 & -5 \end{array}\right]$$

X REF (fails (2)
fails (3))

$$\begin{array}{c} \begin{array}{ccccc} w & x & y & z & \end{array} \\ \left[\begin{array}{ccccc} 1 & 1 & 0 & 3 & -5 \\ 0 & 0 & 1 & -1 & -2 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ \text{pivot} & & \text{pivot} & \end{array} \end{array}$$

✓REF

✓RREF

x, z free variables

↳ give us parameters

↖ This is the augmented matrix for a system of L.E.s:

$$\begin{aligned} w + x \qquad + 3z &= -5 \\ y - z &= -2 \end{aligned}$$

Set the non-free variables (w and y) in terms of the free variables (x and z):

$$w = -5 - x - 3z$$

$$y = -2 + z$$

Now pick parameters for the free variables (e.g. $x = s$
$z = t$)

We get:

$$w = -5 - s - 3t$$
$$x = s$$
$$y = -2 + t$$
$$z = t$$

& so this system has solution $(-5-s-3t, s, -2+t, t)$.

Since we can translate systems of L.E.s into an (augmented) matrix and get exact solution(s) from an RREF matrix, our goal:

Solution algorithm will translate matrices into RREF matrices.

How?

Elementary Row Operations

3 types:

(1) Scale a row by a non-zero constant.

(2) Add a multiple of one row to another.

(3) Swap any two rows.

Important: Applying these to an augmented matrix doesn't change the system's solutions.

Gauss – Jordan Elimination

To solve a system of L.E.s:

Step 1 Write down the augmented matrix of the system.

Step 2 Use elementary row operations to turn the augmented matrix into an RREF matrix.

Step 3 Read off the solution(s) to the system from the RREF matrix.

Remark If we stop partway in Step 2 at a REF matrix this process is called

Gaussian Elimination

Then to find the solution(s) use "back substitution" – we did this implicitly in finding $\left(\frac{9}{7}, \frac{11}{7}\right)$ as solution to

$$\begin{bmatrix} 1 & 3 & 6 \\ 2 & -1 & 1 \end{bmatrix} \text{ augmented matrix } \leftarrow \begin{cases} x + 3y = 6 \\ 2x - y = 1 \end{cases} \longrightarrow \underbrace{\begin{bmatrix} 1 & 3 & 6 \\ 0 & 1 & 11/7 \end{bmatrix}}_{\text{REF}}$$

$\downarrow R_2 \to R_2 - 2R_1$

$$\begin{bmatrix} 1 & 3 & 6 \\ 0 & -7 & -11 \end{bmatrix} \rightarrow$$ This tells us $y = 11/7$; then we got x by: $x = 6 - 3y = 6 - 3\left(\frac{11}{7}\right) = \frac{9}{7}$.

How to do Step 2 (Elimination; matrix → RREF) in practice?

Procedure

Step 1 Augmented matrix.

Step 2 G-J Elimination

Stage I = Gaussian Elim. (to reach REF)

(1) Find left-most non-zero column and make sure its top entry is not zero, swapping rows if necessary.

Example

Solve

$$\begin{aligned} 3y - z &= -2 \\ 2x + 2z &= 1 \\ x + y &= 5 \end{aligned}$$

$$\begin{bmatrix} 0 & 3 & -1 & -2 \\ 2 & 0 & 2 & 1 \\ 1 & 1 & 0 & 5 \end{bmatrix}$$

$\downarrow$ swap R_1 & R_2

$$\begin{bmatrix} 2 & 0 & 2 & 1 \\ 0 & 3 & -1 & -2 \\ 1 & 1 & 0 & 5 \end{bmatrix}$$

$\downarrow R_1 \to \frac{1}{2} R_1$

(2) Multiply through top row to get a leading 1:

$$\begin{bmatrix} 1 & 0 & 1 & 1/2 \\ 0 & 3 & -1 & -2 \\ 1 & 1 & 0 & 5 \end{bmatrix}$$

$R_3 \to R_3 - R_1$

(3) "Kill off" the rest of that row's leading entry column (i.e. make other entries 0) by adding multiples of row 1 to other columns

$$\left[\begin{array}{cccc} 1 & 0 & 1 & 1/2 \\ \hline 0 & 3 & -1 & -2 \\ 0 & 1 & -1 & 9/2 \end{array}\right]$$

(4) Cover up first row & repeat from (1)

(1) top entry of left-most non-zero column

$$\begin{bmatrix} 0 & 3 & -1 & -2 \\ 0 & 1 & -1 & 9/2 \end{bmatrix}$$

(2) set the 3 to be 1

$R_1 \to \frac{1}{3} R_1$

$$\begin{bmatrix} 0 & 1 & -1/3 & -2/3 \\ 0 & 1 & -1 & 9/2 \end{bmatrix}$$

(3) Now kill off → (3) $R_2 \to R_2 - R_1$

Exercise

$$\begin{bmatrix} 0 & 1 & -1/3 & -2/3 \\ 0 & 0 & -2/3 & 31/6 \end{bmatrix}$$

Cover up: $\begin{bmatrix} 0 & 0 & -\frac{2}{3} & \frac{31}{6} \end{bmatrix}$

(2) set to 1: $R_1 \to -\frac{3}{2} R_1$

Exercise

$$\begin{bmatrix} 0 & 0 & 1 & -\frac{31}{4} \end{bmatrix}$$

(3) No other entries to kill off

Stop after last row & uncover rows:

$$\begin{bmatrix} 1 & 0 & 1 & 1/2 \\ 0 & 1 & -1/3 & -2/3 \\ 0 & 0 & 1 & -31/4 \end{bmatrix}$$

Now in REF.

Stage II (REF → RREF)

(1) Add multiples of last non-zero row to

"Kill off" (→ 0) entries above last row's leading 1

$$\begin{bmatrix} 1 & 0 & 0 & 33/4 \\ 0 & 1 & 0 & -13/4 \\ 0 & 0 & 1 & -31/4 \end{bmatrix}$$

(2) If needed, repeat previous step going from right to left, up the rows. $\longrightarrow$ RREF.

(In this example, (2) is not needed as the above matrix is already in RREF.

<u>Step3</u> Now read off solution

$x = 33/4$

$y = -13/4$

$z = -31/4.$