1B03 – LINEAR ALGEBRA 1 (C01) WS19 Lecture 30

Recall A collection $S = \{\bar{v}_1, \dots, \bar{v}_r\}$ of vectors in V

- <u>spans</u> V (or a subspace W) if every vector $\bar{v}$ in V (resp. W) can be written as $k_1\bar{v}_1 + \dots + k_r\bar{v}_r = \bar{v}$.
- is <u>linearly independent</u> if the only choice of scalars $k_1, \dots, k_r$ with $k_1\bar{v}_1 + \dots + k_r\bar{v}_r = \bar{0}$ is $k_1 = \dots = k_r = 0$.

Today Coordinates e.g. $(3,-1,2) = \underline{3}\bar{e}_1 + (\underline{-1})\bar{e}_2 + \underline{2}\bar{e}_3$

<u>Definitions</u> A set of vectors S in a vector space V is a <u>basis</u> for V if (1) $\text{span}(S) = V$

(2) S is linearly independent

If $S = \{\bar{v}_1, \dots, \bar{v}_n\}$ is finite and a basis and $\bar{v} \in V$ is written

$$\bar{v} = c_1\bar{v}_1 + \dots + c_n\bar{v}_n, \quad \text{then}$$

$(c_1, \dots, c_n) = (\bar{v})_S$ is the <u>coordinate vector</u> <u>of $\bar{v}$ relative to</u> (or with respect to) S. Lies in $\mathbb{R}^n$

If we had $c_1\bar{v}_1 + \dots + c_n\bar{v}_n = d_1\bar{v}_1 + \dots + d_n\bar{v}_n = \bar{v}$

i.e. potentially two different coordinate vectors then $(c_1 - d_1)\bar{v}_1 + \dots + (c_n - d_n)\bar{v}_n = \bar{0}$

$\Rightarrow$ $c_i - d_i = 0$ as S is linealy independent.

Examples • $\{\bar{e}_1, \dots, \bar{e}_n\}$ is the standard basis for $\mathbb{R}^n$

Hee $\bar{v} = (v_1, \dots, v_n) = v_1\bar{e}_1 + \dots + v_n\bar{e}_n$

i.e. $v_1, \dots, v_n$ are the coordinates wrt $\bar{e}_1, \dots, \bar{e}_n$ of $\bar{v}$

• $\{1, x, x^2, \dots, x^d\}$ is the standard basis for P_d.
(we already saw it spans P_d & is lin. independent.)
A polynomial $a_0 + a_1 x + a_2 x^2 + \dots + a_d x^d$ has coordinate vector $(a_0, a_1, a_2, \dots, a_d) \in \mathbb{R}^{d+1}$ wrt $\{1, x, \dots, x^d\}$.

• $V = M_{mn}(\mathbb{R})$ has standard basis the $m.n$-many matrices each with a 1 in a different entry & zeros everywhere else.

e.g. for $M_{22}(\mathbb{R})$: $\left\{ \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \right\}$

Example Show that $S = \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \right\}$

is a basis for $\mathbb{R}^3$.

Solution A linear combination of vectors in S looks like

$$c_1 \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} + c_2 \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + c_3 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} 0 & 3 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{\bar{c}} = A\bar{c}$$

- To say "S spans $\mathbb{R}^3$" is to say "every $\bar{b} \in \mathbb{R}^3$ can be written as a linear comb. of vectors in S" i.e. "for every $\bar{b} \in \mathbb{R}^3$ there is a solution $\bar{c} \in \mathbb{R}^3$ to $A\bar{c} = \bar{b}$"

- To say "S lin. independent" is to say "the only solution $\bar{c} \in \mathbb{R}^3$ to $A\bar{c} = \bar{0}$ is $\bar{c} = \bar{0}$"

Since A is square these two statements are equivalent to saying "A invertible"

so check $\det(A) = \begin{vmatrix} 0 & 3 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{vmatrix} = -4 \neq 0$

$\Rightarrow$ A invertible

$\Rightarrow$ S basis.

Example Show that $S = \{1+x,\ x+x^2,\ x^2\}$ is a basis for P_2 and find the coordinates of $5 - 3x + x^2$ relative to S.

Solution Linear comb. of vectors in S:

$$\underbrace{c_1(1+x) + c_2(x+x^2) + c_3 x^2}$$

Set this $= a_0 + a_1 x + a_2 x^2 \cancel{+ a_3 x^3}$

Compare coeffs:

$$c_1 + (c_1+c_2)x + (c_2+c_3)x^2 = a_0 + a_1 x + a_2 x^2$$

i.e.

$$\left.\begin{aligned} c_1 \phantom{{}+c_2+c_3} &= a_0 \\ c_1 + c_2 \phantom{{}+c_3} &= a_1 \\ c_2 + c_3 &= a_2 \end{aligned}\right\} \rightarrow \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} a_0 \\ a_1 \\ a_2 \end{bmatrix}$$

To check S is a basis need to check

- $\text{span}(S) = P_2$ i.e. for any choice of a_is, there <u>is</u> a solution to this system
- S lin. indep. i.e. if $a_i = 0$ for all i, the only solution is $c_j = 0$ for all j

i.e again, since A is square, check if A invertible: $\det(A) = 1 \neq 0$. So yes, S is a basis.

To write $5-3x+2x^2$ relative to S, need to solve above system in this special case i.e.

$$A\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 2 \end{bmatrix}.$$

$\xrightarrow{\text{row reduce}}$ $\left[\begin{array}{ccc|c} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & -8 \\ 0 & 0 & 1 & 10 \end{array}\right]$ i.e. $c_1 = 5$, $c_2 = -8$, $c_3 = 10$

So $5-3x+2x^2 = 5(1+x) - 8(x+x^2) + 10x^2$

& $(5-3x+2x^2)_S = (5, -8, 10)$.

Note In such questions, we end up with A square: explanation to follow in 4.5.

But first:

6.3 Gram-Schmidt Process

Example Find the coordinate vector of $\begin{bmatrix} -1 \\ 7 \\ 7 \end{bmatrix}$ wrt the following basis for $\mathbb{R}^3$:

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} \right\}.$$

Solution Above method: row reduce

$$\left[\begin{array}{ccc|c} 1 & 2 & -1 & -1 \\ 1 & -2 & 1 & 7 \\ 0 & 1 & 4 & 7 \end{array}\right] \rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$$

i.e. $\left(\begin{bmatrix} -1 \\ 7 \\ 7 \end{bmatrix}\right)_S = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ i.e. $\begin{bmatrix} -1 \\ 7 \\ 7 \end{bmatrix} = 3\underline{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}} - \underline{\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}} + 2\underline{\begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}}$

But notice: $\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} = 2 - 2 = 0$

$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} = -1 + 1 = 0$

$\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix} = -2 - 2 + 4 = 0$

<u>Definitions</u> When the vectors in a set $S = \{\bar{v}_1, \dots, \bar{v}_r\}$ satisfy $\bar{v}_j \cdot \bar{v}_i = 0$ for $j \neq i$, we call S an <u>orthogonal set</u>.

- If S is also a basis, we call S an <u>orthogonal basis</u>
- If all $\bar{v}_i$ are unit vectors, we call S an <u>orthonormal set / basis</u>.