1B03 – LINEAR ALGEBRA 1 (C01) WS19 Lecture 31

Last Time Bases & Coordinates

- $S = \{\bar{v}_1, \ldots, \bar{v}_r\}$ is a basis for V if (1) $\text{span}(S) = V$
  (2) S is linearly independent
- If S is a basis for V and $\bar{v} = c_1\bar{v}_1 + \ldots + c_r\bar{v}_r$, then $(\bar{v})_S = (c_1, \ldots, c_n)$ is the vector of coordinates of $\bar{v}$ w.r.t. S
- S in $\mathbb{R}^n$ is an orthogonal set/basis if $\bar{v}_j \cdot \bar{v}_i = 0$ for $j \neq i$ & orthonormal if (a) it is orthogonal & (b) $\|\bar{v}_i\| = 1$ for all i.

Example To find coord. vector of $\begin{bmatrix} -1 \\ 7 \\ 7 \end{bmatrix}$ wrt basis

$$S = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 4 \end{bmatrix} \right\}$$

$\rightarrow$ solve $\left[\begin{array}{ccc|c} 1 & 2 & 1 & -1 \\ 1 & -2 & 1 & 7 \\ 0 & 1 & 4 & 7 \end{array}\right]$

$\rightsquigarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 2 \end{array}\right]$ so $\left(\begin{bmatrix} -1 \\ 7 \\ 7 \end{bmatrix}\right)_S = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$.

Say $S = \{\bar{v}_1, \ldots, \bar{v}_r\}$ is an orthogonal basis for $\mathbb{R}^n$ and $\bar{v} \in \mathbb{R}^n$ is $\bar{v} = c_1\bar{v}_1 + \ldots + c_r\bar{v}_r$

Then for each $\bar{v}_i$ we have:

$$\underline{\bar{v}_i \cdot \bar{v}} = \bar{v}_i \cdot (c_1 \bar{v}_1 + \ldots + c_r \bar{v}_r)$$

$$= c_1 \bar{v}_i \cdot \bar{v}_1 + \ldots + c_r \bar{v}_i \cdot \bar{v}_r$$

$$= c_i \bar{v}_i \cdot \bar{v}_i = \underline{c_i \|\bar{v}_i\|^2} \quad \text{as } S \text{ orthogonal}$$

$$\Rightarrow c_i = \frac{\bar{v}_i \cdot \bar{v}}{\|\bar{v}_i\|^2} \leftarrow = 1 \text{ if } S \text{ orthonormal}$$

for each $i = 1, \ldots, r$

So our example revisited:

coord. vector of $\underbrace{\begin{bmatrix} -1 \\ 7 \\ 7 \end{bmatrix}}_{\bar{v}}$ wrt $S = \left\{ \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}_{\bar{v}_1}, \underbrace{\begin{bmatrix} 2 \\ -2 \\ 1 \end{bmatrix}}_{\bar{v}_2}, \underbrace{\begin{bmatrix} -1 \\ 1 \\ 4 \end{bmatrix}}_{\bar{v}_3} \right\}$

$$c_1 = \frac{\bar{v}_1 \cdot \bar{v}}{\|\bar{v}_1\|^2} = \frac{-1+7}{1+1} = \frac{6}{2} = 3 \qquad c_2 = \frac{\bar{v}_2 \cdot \bar{v}}{\|\bar{v}_2\|^2} = \frac{-2-14+7}{4+4+1} = \frac{-9}{9} = -1$$

$$c_3 = \frac{\bar{v}_3 \cdot \bar{v}}{\|\bar{v}_3\|^2} = \frac{1+7+28}{1+1+16} = \frac{36}{18} = 2.$$

Remember, this trick <u>ONLY</u> works if <u>S orthogonal</u>!

<u>Fact</u> If $S = \{\bar{v}_1, \ldots, \bar{v}_r\}$ is orthogonal & $\bar{v}_i \neq \bar{0}$ in $\mathbb{R}^n$, then S is linearly independent.

Why? As always set $\bar{0} = k_1\bar{v}_1 + \ldots + k_r\bar{v}_r$

Similar to before, for each $\bar{v}_i$:

$$0 = \bar{0}\cdot\bar{v}_i = (k_1\bar{v}_1 + \ldots + k_r\bar{v}_r)\cdot\bar{v}_i$$
$$= k_1\bar{v}_1\cdot\bar{v}_i + \ldots + k_r\bar{v}_r\cdot\bar{v}_i$$
$$= k_i\bar{v}_i\cdot\bar{v}_i \quad (\text{as } S \text{ orthogonal})$$
$$= k_i\|\bar{v}_i\|^2$$

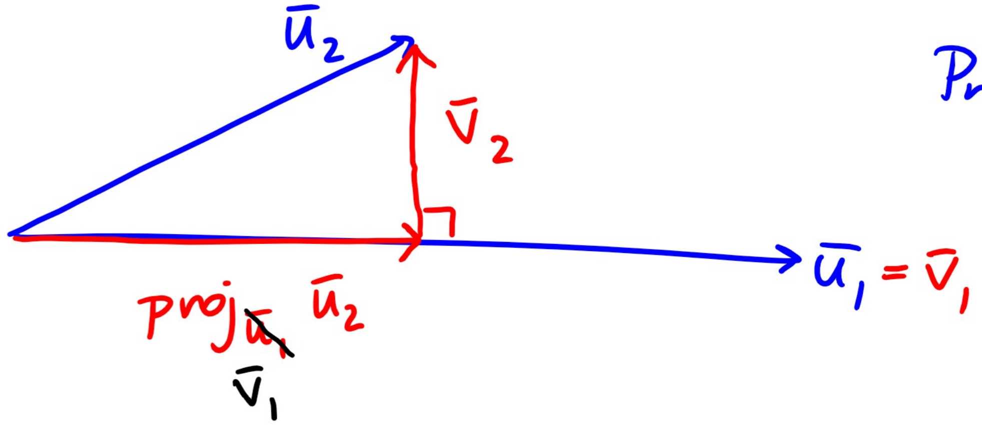
$\Rightarrow k_i = 0$ (as $\|\bar{v}_i\| \neq 0$). for each $i = 1, \ldots, r$.

So S linearly independent.

How to get an orthogonal basis from any basis

In $\mathbb{R}^3$, $S = \{\bar{u}_1, \bar{u}_2\}$ if, $\bar{u}_1, \bar{u}_2$ NOT colinear, (This says S linearly independent, hence a basis for its span.)

then S is a basis for span(S) i.e. a plane W



Projection Theorem:

$$\bar{u}_2 = \text{proj}_{\cancel{\bar{u}_1}\,\bar{v}_1}\bar{u}_2 + \bar{v}_2$$

where $\bar{v}_2\cdot\bar{u}_1 = 0$

Fact Span$\{\bar{v}_1, \bar{v}_2\}$ = span$\{\bar{u}_1, \bar{u}_2\} = W$, and (we see $\{\bar{v}_1, \bar{v}_2\}$ is an orthogonal basis for W,

where $\bar{v}_1 = \bar{u}_1$ & $\bar{v}_2 = \bar{u}_2 - \text{proj}_{\bar{v}_1} \bar{u}_2$

$$\Rightarrow \bar{v}_2 = \bar{u}_2 - \frac{\bar{u}_2 \cdot \bar{v}_1}{\|\bar{v}_1\|^2} \bar{v}_1$$

Take away any part of $\bar{u}_2$ coming from $\bar{v}_1$.

<u>Example</u> Find an orthogonal basis for for the plane spanned by

$$\left\{ \underbrace{\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}}_{\bar{u}_1}, \underbrace{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}}_{\bar{u}_2} \right\}.$$

<u>Solution</u> $\bar{v}_1 = \bar{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$.

$$\bar{v}_2 = \bar{u}_2 - \text{proj}_{\bar{v}_1} \bar{u}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}}{\left\| \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \right\|^2} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}$$

$$= \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{1}{11} \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} 21/11 \\ 12/11 \\ -3/11 \end{bmatrix}.$$

So orthogonal basis is $\left\{ \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 21/11 \\ 12/11 \\ -3/11 \end{bmatrix} \right\}$

& notice $\bar{v}_1 \cdot \bar{v}_2 = \frac{21}{11} - \frac{12}{11} - \frac{9}{11} = 0$.

Gram-Schmidt Procedure

$\rightarrow$ above idea with maybe more vectors.

$S = \{\bar{u}_1, \dots, \bar{u}_r\}$ - basis for subspace W of $\mathbb{R}^n$

The following process produces an orthogonal basis $\{\bar{v}_1, \dots, \bar{v}_r\}$ for W with

$$\text{Span}(\{\bar{u}_1, \dots, \bar{u}_k\}) = \text{span}(\{\bar{v}_1, \dots, \bar{v}_k\})$$

for every $1 \leq k \leq r$.

$$\bar{v}_1 = \bar{u}_1$$

Take away the part of $\bar{u}_2$ already coming from $\bar{v}_1$ $\rightarrow$ left with the part of $\bar{u}_2$ orthogonal to $\bar{v}_1$.

$$\bar{v}_2 = \bar{u}_2 \overbrace{- \text{proj}_{\bar{v}_1} \bar{u}_2} = \bar{u}_2 - \frac{\bar{u}_2 \cdot \bar{v}_1}{\|\bar{v}_1\|^2} \bar{v}_1$$

Take away the part of $\bar{u}_3$ already coming from the new vectors so far on the list i.e. coming from $\bar{v}_1, \bar{v}_2$.

- left only with the part of $\bar{u}_3$ orthogonal to all of $\bar{v}_1, \bar{v}_2$

$$\bar{v}_3 = \bar{u}_3 \overbrace{- \text{proj}_{\bar{v}_1} \bar{u}_3 - \text{proj}_{\bar{v}_2} \bar{u}_3}$$

$$= \bar{u}_3 - \frac{\bar{u}_3 \cdot \bar{v}_1}{\|\bar{v}_1\|^2} \bar{v}_1 - \frac{\bar{u}_3 \cdot \bar{v}_2}{\|\bar{v}_2\|^2} \bar{v}_2$$

$\vdots$

Take away the part of $\bar{u}_r$ coming from vectors already on list: $\bar{v}_1, \dots, \bar{v}_{r-1}$ & left with part of $\bar{u}_r$ orthogonal to $\bar{v}_1, \dots, \bar{v}_r$.

$$\bar{v}_r = \bar{u}_r \overbrace{- \text{proj}_{\bar{v}_1} \bar{u}_r - \text{proj}_{\bar{v}_2} \bar{u}_r - \dots - \text{proj}_{\bar{v}_{r-1}} \bar{u}_r}$$

To get an orthonormal basis, 2 options:

① Replace $\{\bar{v}_1, \ldots, \bar{v}_r\}$ with $\left\{\frac{\bar{v}_1}{\|\bar{v}_1\|}, \ldots, \frac{\bar{v}_r}{\|\bar{v}_r\|}\right\}$

at the end of the process.

② At every stage, normalize, & then work with the normalized vector going forward.

<u>Example</u> Find an orthonormal basis for

$$\text{span}\{\underbrace{(1,0,1)}_{\bar{u}_1}, \underbrace{(-2,5,-2)}_{\bar{u}_2}, \underbrace{(4,-10,2)}_{\bar{u}_3}\}$$

$\bar{v}_1 = \bar{u}_1 = (1,0,1).$

$$\bar{v}_2 = \bar{u}_2 - \text{proj}_{\bar{v}_1}\bar{u}_2 = (-2,5,-2) - \frac{\bar{u}_2 \cdot \bar{v}_1}{\|\bar{v}_1\|^2}\bar{v}_1$$

$$= (-2,5,-2) - \frac{-4}{2}(1,0,1)$$

$$= (-2,5,-2) + 2(1,0,1)$$

$$= (0,5,0).$$

$$\bar{v}_3 = \bar{u}_3 - \text{proj}_{\bar{v}_1}\bar{u}_3 - \text{proj}_{\bar{v}_2}\bar{u}_3$$

$$= (4,-10,2) - \underbrace{\frac{\overbrace{(4,-10,2)\cdot(1,0,1)}^{6}}{2}}_{3}(1,0,1)$$

$$-\underbrace{\frac{\overbrace{(4,-10,2)\cdot(0,5,0)}^{-50}}{25}}_{-2}(0,5,0) = (1,0,-1).$$

$$= (4,-10,2) - 3(1,0,1) + 2(0,5,0)$$

So an orthogonal basis for $\text{span}\left(\{(1,0,1), (-2,5,-2), (4,-10,2)\}\right)$

is $\{(1,0,1), (0,5,0), (1,0,-1)\}$.

And an orthonormal basis is found using $\|(1,0,1)\| = \sqrt{1+1} = \sqrt{2}$

$$\|(0,5,0)\| = \sqrt{5^2} = 5$$

$$\|(1,0,-1)\| = \sqrt{1+1} = \sqrt{2};$$

we get $\left\{\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), (0,1,0), \left(\frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}}\right)\right\}$.

Method 2 gives:

Method 1 gives $\bar{v}_1 = \bar{u}_1 = (1,0,1)$.

Method 2 gives $\bar{v}_1 = \dfrac{(1,0,1)}{\|(1,0,1)\|} = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$.

Now use $\bar{v}_1 = \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$ in later steps.

Method 1 now gives $\bar{v}_2 = (-2,5,-2) - \dfrac{(-2,5,-2)\cdot\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)}{\left\|\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)\right\|^2}\left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right)$

But of course this is 1.

$$= (-2,5,2) - (-\sqrt{2}-\sqrt{2})\left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)$$

$$= (-2,5,2) + (2,0,2) = (0,5,0).$$

Now method ② gives $\bar{v}_2 = \dfrac{(0,5,0)}{\|(0,5,0)\|} = (0,1,0).$

We proceed using $\bar{v}_1 = \left(\frac{1}{\sqrt{2}},0,\frac{1}{\sqrt{2}}\right)$ and $\bar{v}_2 = (0,1,0)$.

$$\bar{v}_3 = (4,-10,2) - \left[(4,-10,2)\cdot\left(\tfrac{1}{\sqrt{2}},0,\tfrac{1}{\sqrt{2}}\right)\right]\left(\tfrac{1}{\sqrt{2}},0,\tfrac{1}{\sqrt{2}}\right)$$

divide by 1 →

$$- \left[(4,-10,2)\cdot(0,1,0)\right](0,1,0)$$

divide by 1 →

$$= (4,-10,2) - \underbrace{(2\sqrt{2}+\sqrt{2})\left(\tfrac{1}{\sqrt{2}},0,\tfrac{1}{\sqrt{2}}\right)} + \underbrace{10(0,1,0)}$$

$$= (4,-10,2) - (3,0,3) + (0,10,0)$$

$$= (1,0,-1).$$

Method ② now gives $\bar{v}_3 = \dfrac{(1,0,-1)}{\|(1,0,-1)\|} = \left(\frac{1}{\sqrt{2}},0,-\frac{1}{\sqrt{2}}\right).$

So we end up with the same orthonormal basis, and some parts of the calculations were harder, but some were easier. You do this if you want an orthonormal basis at each step.