1B03 – LINEAR ALGEBRA 1 (C01) WS19 Lecture 32

Yesterday Gram–Schmidt Process

How to get an orthogonal basis for subspace W in $\mathbb{R}^n$ from any basis for W.

Old basis : $\{\bar{u}_1, \ldots, \bar{u}_r\}$

New basis : $\{\bar{v}_1, \ldots, \bar{v}_r\}$ with $\text{span}(\{\bar{u}_1, \ldots, \bar{u}_k\}) = \text{span}(\{\bar{v}_1, \ldots, \bar{v}_k\})$ for $1 \le k \le r$

$\hookrightarrow \bar{v}_1 = \bar{u}_1$

$\vdots$

$$\bar{v}_n = \bar{u}_n - \overbrace{\frac{\bar{u}_n \cdot \bar{v}_1}{\|\bar{v}_1\|^2}\bar{v}_1}^{\text{proj}_{\bar{v}_1}\bar{u}_n} - \overbrace{\frac{\bar{u}_n \cdot \bar{v}_2}{\|\bar{v}_2\|^2}\bar{v}_2}^{\text{proj}_{\bar{v}_2}\bar{u}_n} - \cdots - \overbrace{\frac{\bar{u}_n \cdot \bar{v}_{n-1}}{\|\bar{v}_{n-1}\|^2}\bar{v}_{n-1}}^{\text{proj}_{\bar{v}_{n-1}}\bar{u}_n}$$

$\vdots$

$\bar{v}_r$.

To get an orthonormal basis : $\left\{\frac{v_1}{\|v_1\|}, \ldots, \frac{v_r}{\|v_r\|}\right\}$.

Example Find an orthonormal basis for

$$W = \text{span}(\{\underbrace{(1,0,1)}_{\bar{u}_1}, \underbrace{(0,5,-2)}_{\bar{u}_2}, \underbrace{(4,-3,2)}_{\bar{u}_3}\}).$$

$\bar{v}_1 = \bar{u}_1 = (1,0,1) \longrightarrow$ norm : $\|(1,0,1)\| = \sqrt{1+1} = \sqrt{2}$

$$\bar{v}_2 = \bar{u}_2 - \text{proj}_{\bar{v}_1}\bar{u}_2 = (0,5,-2) - \frac{(0,5,-2)\cdot(1,0,1)}{\|(1,0,1)\|^2}(1,0,1)$$

$$= (0,5,-2) \underbrace{- \frac{-2}{2}}_{+} (1,0,1) = (1,5,-1)$$

Norm : $\|(1,5,-1)\| = \sqrt{1+25+1} = \sqrt{27} = 3\sqrt{3}$

$$\bar{v}_3 = \bar{u}_3 - \text{proj}_{\bar{v}_1}\bar{u}_3 - \text{proj}_{\bar{v}_2}\bar{u}_3$$

$$= (4,-3,2) - \frac{(4,-3,2)\cdot(1,0,1)}{\|(1,0,1)\|^2}(1,0,1) - \frac{(4,-3,2)\cdot(1,5,-1)}{\|(1,5,-1)\|^2}(1,5,-1)$$

$= (4,-3,2) - \frac{6}{2}(1,0,1) - \frac{4-15-2}{27}(1,5,-1)$

$= \left(\frac{40}{27}, -\frac{16}{27}, -\frac{40}{27}\right) \rightarrow$ Norm $\left\|\left(\frac{40}{27}, -\frac{16}{27}, -\frac{40}{27}\right)\right\|$

$= \frac{\sqrt{3456}}{27}$

So orthogonal basis:

$\left\{(1,0,1), (1,5,-1), \left(\frac{40}{27}, -\frac{16}{27}, -\frac{40}{27}\right)\right\}$

So orthonormal basis:

$\left\{ \left(\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}\right), \left(\frac{1}{3\sqrt{3}}, \frac{5}{3\sqrt{3}}, -\frac{1}{3\sqrt{3}}\right), \left(\frac{40}{\sqrt{3456}}, \frac{-16}{\sqrt{3456}}, \frac{-40}{\sqrt{3456}}\right) \right\}$.

This is really a special use of:

Souped-Up Projection Theorem If $\bar{u} \in \mathbb{R}^n$ and W is a subspace of $\mathbb{R}^n$ then we can uniquely write $\bar{u} = \bar{w}_1 + \bar{w}_2$ where $\bar{w}_1 \in W$ & $\bar{w}_2 \cdot \bar{w} = 0$ for all $\bar{w} \in W$

(Earlier Proj. Thm: special case $W = \text{span}(\{\bar{a}\})$.)

If $\{\bar{v}_1, \dots, \bar{v}_r\}$ is an orthogonal basis for W then $\bar{w}_1$ – we call this $\text{proj}_W \bar{u}$ – is:

$$\text{proj}_W \bar{u} = \text{proj}_{\bar{v}_1} \bar{u} + \text{proj}_{\bar{v}_2} \bar{u} + \ldots + \text{proj}_{\bar{v}_r} \bar{u}$$

(So Gram-Schmidt says: $\bar{v}_1 = \bar{u}_1$

$\vdots$

each $\bar{v}_n = \bar{u}_n - \text{proj}_{\text{Span}\{\bar{v}_1, \ldots, \bar{v}_{n-1}\}} \bar{u}_n$.

$\vdots$)

<u>Example</u> Find the projection of $(3,-2,7)$ onto the xy-plane.

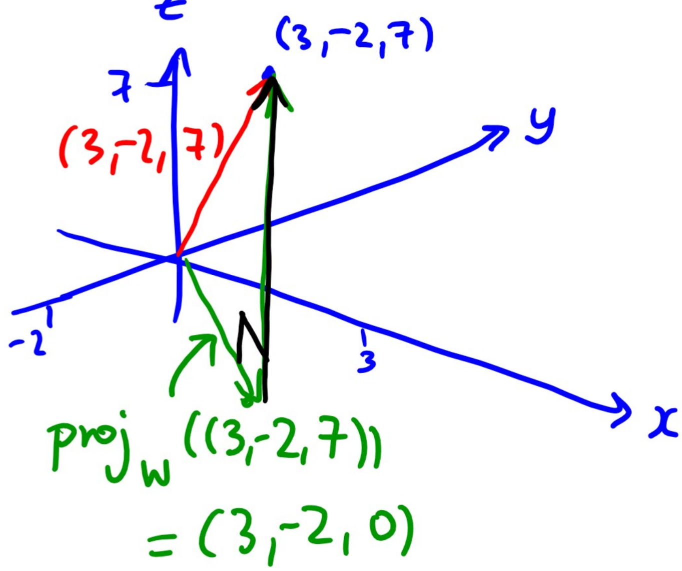
<u>Solution</u> $W = xy\text{-plane} = \text{span}(\{\underset{\uparrow}{\overset{(1,0,0)}{\overset{\shortparallel}{\bar{e}_1}}}, \underset{\nearrow}{\overset{(0,1,0)}{\overset{\shortparallel}{\bar{e}_2}}}\})$

check: $\bar{e}_1 \cdot \bar{e}_2 = 0$

So $\text{proj}_W((3,-2,7)) =$

$$((3,-2,7)\cdot\bar{e}_1)\,\bar{e}_1 + ((3,-2,7)\cdot\bar{e}_2)\,\bar{e}_2$$

$$= (3,-2,0)$$



<u>Example</u> Find the projection of $\bar{u} = \begin{bmatrix} 27 \\ 2 \\ -1 \end{bmatrix}$ onto $W = \text{span}\left\{\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}\right\}$

<u>Solution</u> Is $\nearrow$ orthogonal!?

No! $\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = 1 \neq 0$

First we need an orthogonal basis for W.

We did this last time using Gram-Schmidt:

$$W = \text{span}\left\{ \underbrace{\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix}}_{\bar{v}_1}, \underbrace{\begin{bmatrix} 21/11 \\ 12/11 \\ -3/11 \end{bmatrix}}_{\bar{v}_2} \right\}.$$

$$\bar{u} = \begin{bmatrix} 27 \\ 2 \\ -1 \end{bmatrix}$$

Now
$$\text{proj}_W \bar{u} = \frac{\bar{u}\cdot\bar{v}_1}{\|\bar{v}_1\|^2}\bar{v}_1 + \frac{\bar{u}\cdot\bar{v}_2}{\|\bar{v}_2\|^2}\bar{v}_2$$

$$= \frac{27-2-3}{11}\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + \frac{\frac{(21)(27)}{11}+\frac{24}{11}+\frac{3}{11}}{\frac{21^2+12^2+(-3)^2}{11^2}}\begin{bmatrix} 21/11 \\ 12/11 \\ -3/11 \end{bmatrix}$$

2   11

$$= 2\begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} + \begin{bmatrix} 21 \\ 12 \\ -3 \end{bmatrix} = \begin{bmatrix} 23 \\ 10 \\ 3 \end{bmatrix}.$$

4.5 Dimension

Def^n Vector space V is <u>finite dimensional</u> if it has a finite basis.

Otherwise V <u>infinite dimensional</u>.

Theorem If V has a basis $S = \{\bar{v}_1, \dots, \bar{v}_n\}$ then (1) any set with $> n$ vectors is linearly dependent

(2) any set with $< n$ vectors does NOT span V.

So if V finite dimensional, then every basis for V has the same # of elements, called <u>dimension of V</u>, $\dim(V)$.

(<u>Note</u>: The space $\{\bar{0}\}$ has no basis — so $\dim = 0$.)

sometimes people say that the empty set is the only basis for $\{\bar{0}\}$

<u>Examples</u>

- $\mathbb{R}^n$ has e.g. basis $\{\bar{e}_1, \ldots, \bar{e}_n\}$ so $\dim(\mathbb{R}^n) = n$
- P_d has e.g. basis $\{1, x, x^2, \ldots, x^d\}$ so $\dim(P_d) = d+1$
- $M_{mn}(\mathbb{R})$ has e.g. basis $\left\{ \begin{bmatrix} 1 & 0 & \cdots \\ 0 & 0 & \\ \vdots & & \ddots \end{bmatrix}, \ldots \begin{bmatrix} \ddots & & \vdots \\ & 0 & 0 \\ \cdots & 0 & 1 \end{bmatrix} \right\}$

  so $\dim(M_{mn}(\mathbb{R})) = mn$.

Very useful facts:

<u>$\pm$ Plus/Minus Theorem</u> $\quad S$ non-empty set of vectors in V

(a) S lin. independent but $\operatorname{span}(S) \neq V$, then for any $\bar{v} \notin \operatorname{span}(S)$, $\underbrace{S \cup \{\bar{v}\}}_{\text{add } \bar{v} \text{ to } S}$ is also lin. independent.

(b) S lin. dependent, & $\bar{v} \in S$ a linear comb. of other vectors in S, then $\operatorname{span}(S) = \operatorname{span}(\underbrace{S - \{\bar{v}\}}_{\text{delete } \bar{v} \text{ from } S})$

<u>Theorem</u> If $\dim(V) = n$ and a set S has n vectors, then

S is a basis $\iff$ S is linearly independent OR $\operatorname{span}(S) = V$

inclusive or (i.e. could - & will here - be both!)

So: to check that a set S is a basis for V, first <u>count</u> the # of vectors in V. Does it equal $\dim(V)$?

If no, S cannot be a basis for V.

If yes, still need to check if S is a basis for V or not, but the Theorem here says that (after counting) it is enough to check linear independence OR $\operatorname{span}(S) = V$.

If one of those is true and # vectors in S = $\dim(V)$, then S is a basis for V (i.e. the only one is also true).