1B03 – LINEAR ALGEBRA 1 (C01) WS19 Lecture 33

Last Time Dimension

V – vector space (could be a subspace)

S – $\{\bar{v}_1, \dots, \bar{v}_n\}$ basis for V

every basis for V has same #

dim(V) = dimension of V = n = # elements in S

If dim(V) = n, then
any set with > n elements is linearly DEPENDENT
any set with < n elements does NOT span V

Theorem If dim(V) = n, and $S \subseteq V$ has n vectors, then S is a basis iff S is linearly independent OR S spans V

Example Which of the following are bases for $\mathbb{R}^3$?

$S_1 = \{(1,0,0), (0,1,2)\}$

$S_2 = \{(3,2,1), (-1,0,1), (1,4,7)\}$

$S_3 = \{(1,0,0), (-1,1,0), (0,0,1)\}$

$S_4 = \{(1,0,1), (0,1,0), (1,0,-1), (0,0,1)\}$

First: count! We know $\dim(\mathbb{R}^3) = 3$ so every basis has 3 elements $\longrightarrow$ so S_1, S_4 NOT bases for $\mathbb{R}^3$.

To check S_2, S_3, Theorem says enough to check <u>one</u> of linear independence <u>or</u> spanning. ($\text{span}(S_i) = \mathbb{R}^3$)

Earlier method: write matrix with vectors as columns

→ we saw: invertible iff lin. independent iff set spans

check: det $\neq 0$?

(so, in other words, this is the matrix way of saying "if we have the right # of elements, lin. ind. ⇔ spans")

So for S_2: $\det \begin{bmatrix} 3 & -1 & 1 \\ 2 & 0 & 4 \\ 1 & 1 & 7 \end{bmatrix} = -4 - 10 + 7(14) = 0$

so matrix NOT invertible

So S_2 not a basis.

So for S_3: $\det \begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 1 \neq 0$

So matrix <u>is</u> invertible

so S_3 is a basis.

<u>Example</u> Which of the following are bases for P_2?

$T_1 = \{x + x^2, 1 + x\}$ X not enough elements → $\dim(P_2) = 3$

$T_2 = \{1 - x, 1 + x^2, x + x^2\}$

$T_3 = \{1, x + x^2, 1 + x^2\}$

What about T_2, T_3?

T_2: Notice $(1-x) - (1+x^2) = -x - x^2$
$= -(x + x^2)$

So T_2 NOT linearly independent hence NOT a basis for P_2

As for T_3 : suppose $k_1(1) + k_2(x+x^2) + k_3(1+x^2) = 0$

i.e. $(k_1+k_3) + k_2 x + (k_2+k_3)x^2 = 0$

$$\Leftrightarrow \begin{cases} k_1 + k_3 = 0 & \Rightarrow k_1 = 0. \\ k_2 = 0 & \\ k_2 + k_3 = 0 & \Rightarrow k_3 = 0 \end{cases}$$

So T_3 is linearly independent, and hence by the Theorem (since T_3 has 3 elements) T_3 is a basis for P_2.

Other consequences of the Theorem

If V finite-dimensional, S finite set of vectors in V,

then (1) if $\text{span}(S) = V$ but S not a basis, we can throw out vectors from S to get a basis

(2) if S is lin. independent but $\text{span}(S) \neq V$ then we can add vectors to S to get a basis

And if W is a subspace of V

(3) $\dim(W) \leq \dim(V)$

(4) $\dim(W) = \dim(V)$ iff $W = V$.

Example Find a basis for

$$V = \text{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -2 \\ 3 \\ 1 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ -2 \\ 0 \end{bmatrix}, \begin{bmatrix} -2 \\ -2 \\ 0 \\ 1 \end{bmatrix} \right\}$$

<u>Solution</u> First eyeball it $\begin{bmatrix}-2\\0\\-2\\0\end{bmatrix} = -2\begin{bmatrix}1\\0\\1\\0\end{bmatrix}$

So $\text{span}\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-2\\3\\1\end{bmatrix}, \begin{bmatrix}-2\\-2\\0\\1\end{bmatrix}\right\} = V$

$$-3\begin{bmatrix}1\\0\\1\\0\end{bmatrix} + \begin{bmatrix}1\\-2\\3\\1\end{bmatrix} = \begin{bmatrix}-2\\-2\\0\\1\end{bmatrix}$$

So $\text{span}\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-2\\3\\1\end{bmatrix}\right\} = V$

Now check that $\left\{\begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-2\\3\\1\end{bmatrix}\right\}$ is linearly independent. (<u>Exercise</u>)

$$\left[\begin{array}{cc|c}1&1&0\\0&-2&0\\1&3&0\\0&1&0\end{array}\right] \to \left[\begin{array}{cc|c}1&1&0\\0&-2&0\\0&2&0\\0&1&0\end{array}\right] \to \left[\begin{array}{cc|c}1&1&0\\0&1&0\\0&0&0\\0&0&0\end{array}\right] \to \left[\begin{array}{cc|c}1&0&0\\0&1&0\\0&0&0\\0&0&0\end{array}\right] \checkmark$$

<u>Example</u> Find the dimension of

$$W = \{a_0 + a_1x + a_2x^2 \text{ with } a_0 - 3a_1 = a_2\}$$

<u>Exercise</u> Check W <u>is</u> a subspace of P_2.

① $1 + x - 2x^2 \in W$ ✓

② $a_0 + a_1x + (a_0 - 3a_1)x^2 + b_0 + b_1x + (b_0 - 3b_1)x^2 = (a_0+b_0) + (a_1+b_1)x + (a_0+b_0 - 3(a_1+b_1))x^2$ ✓

③ $k(a_0 + a_1x + (a_0 - 3a_1)x^2) = ka_0 + (ka_1)x + (ka_0 - 3(ka_1))x^2$ ✓

So $\dim(W) \le \dim(P_2) = 3$

We want a basis for W so we can count its size.

$1 + x - 2x^2 \in W$ so $W \neq \{0\}$ so $\dim W > 0$ (by ③, ④ above)

but also $1 + x + x^2 \notin W$ so $W \neq P_2$

then $\dim W < 3$ by ③, ④ above.

So dim $W = 1$ or 2.

Now we know if we find a linearly independent set which spans W we have a basis | AND it will have one or two elements.

this part we always knew by the definition of basis!

2 approaches: (1) Try to translate condition on W into a basis:

Look at poly. in W: $a_0 + a_1 x + (a_0 - 3a_1)x^2$

$$a_0(1+x^2) + a_1(x-3x^2)$$

So $\operatorname{span}\{1+x^2, x-3x^2\} = W$

Now check linear independence:

$$\text{set } 0 = k_1(1+x^2) + k_2(x-3x^2)$$
$$= \underbrace{k_1} + \underbrace{k_2}x + (k_1 - 3k_2)x^2$$
$$\Rightarrow k_1 = 0 = k_2$$

So dim $W = 2$.

But any choice of 2 linearly indep. spanning vectors of W would do:

(2) see below

e.g. $\{1 + x - 2x^2, x - 3x^2\}$

Wait, what just happened? Where did this come from?

Well, we know that $0 < \dim(W) < 3$ i.e

$\dim(W) = 1$ or 2, so

we're looking for a basis with 1 or 2 elements.

We also know, by consequence (1) of the Theorem (see p. 3 above), that we can start with anything and expand it to a basis (if it isn't a basis already, in which case we're already done).

So I took $1 + x - 2x^2$, which I already gave above as an example of a vector in W.

Does this span W? i.e. is every polynomial with $a_0 - 3a_1 = a_2$ a linear multiple/combination of $1 + x - 2x^2$?

Well, no. Here's an example of something that is in W but not a linear multiple of $1 + x - 2x^2$:

$$x - 3x^2.$$

So what if I take $\{1 + x - 2x^2, x - 3x^2\}$?

It's linearly independent by the +/− Theorem from last time, so it's a basis for its own span.

Is $\mathrm{span}(\{1+x-2x^2, x-3x^2\}) = W$? Well, this span is a vector space of dimension 2 (it has a basis of 2 vectors) living inside W, so $\dim(W) \geq 2$, and we know from above that $\dim(W) < 3$, so $\dim(W) = 2$. Thus $\mathrm{span}(\{1+x-2x^2, x-3x^2\}) = W$, since the only subspace of W with the same dimension as W is W itself. If you don't like that, check that it spans:

$$a_0 + a_1x + a_2x^2 = b_1(1+x-2x^2) + b_2(x-3x^2) \Rightarrow \begin{array}{l} b_1 = a_0 \\ b_2 = a_1 - b_1 = a_1 - a_0 \end{array} \ \& \ b_3 = \frac{a_2+2b_1}{-3} = \frac{a_2}{-3} - \frac{2a_0}{3} \checkmark.$$

<u>Exerise</u> Find a basis for $\{A \in M_{33}(\mathbb{R})$ with $tr(A) = 0\}$.

(i.e. the set of 3×3 matrices A with $tr(A) = 0$.)