1B03 – LINEAR ALGEBRA 1 (C01) WS19 — Lecture 34

Last Time — Consequences of the ± Theorem

V finite-dimensional $\quad S$ finite set of vectors in V

W subspace of V

- If $\text{span}(S) = V$ but S too big to be a basis for V, we can throw out any vector that is a linear combination of any others until we get a basis for V
- If S linearly independent but too small to be a basis for V, we can keep adding vectors not in the span until we get a basis for V.
- $\dim(W) \leq \dim(V)$ & $\dim(W) = \dim(V)$ iff $W = V$.

4.7 Row Space, Column Space & Null Space

... of A

$$A = \begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & & & \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix} \begin{matrix} \leftarrow \bar{r}_1 \\ \leftarrow \bar{r}_2 \\ \vdots \\ \leftarrow \bar{r}_m \end{matrix}$$

$$\underbrace{\bar{c}_1 \quad \bar{c}_2 \quad \cdots \quad \bar{c}_n}_{\text{Column vectors of } A}$$

Row vectors of A

Row space of A

$\text{row}(A) = \text{Span}(\{\bar{r}_1, \dots, \bar{r}_m\})$ – subspace of $\mathbb{R}^n$

Column space $\text{col}(A) = \text{span}(\{\bar{c}_1, \dots, \bar{c}_n\})$ – subspace of $\mathbb{R}^m$

Recall $\text{null}(A) = \{\bar{x} \in \mathbb{R}^n \text{ with } A\bar{x} = \bar{0}\}$

— solution space to $A\bar{x} = \bar{0}$

Recall From Lecture 24 (3.4) $\rightarrow$ if $A\bar{x} = \bar{b}$ is a system of LEs

every solution $\bar{x} = \bar{x}_0 + \bar{w}$ in null(A) i.e. $A\bar{w} = \bar{0}$

particular solution

We can rewrite this in terms of $\underbrace{\text{a basis for null}(A)}$:

Given $S = \{\bar{v}_1, \dots, \bar{v}_\ell\}$

& $\bar{b} \in \mathbb{R}^m$ & particular soln $\bar{x}_0$ to $A\bar{x} = \bar{b}$

we have: solution set for $A\bar{x} = \bar{b}$ looks like:

$$\{\bar{x}_0 + k_1\bar{v}_1 + \dots + k_\ell \bar{v}_\ell \text{ , for } k_1, \dots, k_\ell \in \mathbb{R}\}$$

How to find a basis for null(A) ?

Solve $A\bar{x} = \bar{0}$: Basis for soln space = basis for null(A) !

same things!!

Recall We can write $A\bar{x} = \underbrace{x_1\bar{c}_1 + x_2\bar{c}_2 + \dots + x_n\bar{c}_n}_{\in \text{col}(A)}$

So to say $A\bar{x} = \bar{b}$ is to say $\bar{b} \in \text{col}(A)$ *

& if $\bar{b} \in \text{col}(A)$ then it is a solution to $A\bar{x} = \bar{b}$

column vectors

$\bar{x}$ = coefficients of columns

*so to find $\bar{b}$ as a linear combination of columns of A, solve $A\bar{x}=\bar{b}$

How to find bases for col(A) (and row(A))?

Use <u>Key Facts</u>

If A can be transformed into B using an ERO

← Elementary Row Operations

then

(i) $\text{null}(A) = \text{null}(B)$ ← we know this! EROs don't change soln space to $A\bar{x}=\bar{0}$

(ii) $\text{row}(A) = \text{row}(B)$ ← row ops don't change span of the rows!

(iii) the dependence relations amongst the columns of A are the same as the dependence relations amongst the columns of B

Also it's easy to see what's going with an RREF matrix.

Example $A = \begin{bmatrix} 1 & 4 & 2 & -6 & -1 \\ 0 & 1 & 0 & -1 & -1 \\ 3 & 14 & 6 & -20 & -4 \\ -1 & -1 & -2 & 3 & 3 \end{bmatrix}$. Find bases for null(A), row(A), col(A)

& write the dependence relations between columns of A.

Solution Take $A \to$ RREF $\begin{bmatrix} 1 & 0 & 2 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} = R$

$x_3 = s$ $x_4 = t$

To find basis for null(A),

solve $A\bar{x} = \bar{0}$ i.e. row reduce $[A \mid 0] \to [R \mid 0]$

$x_1 + 2s - 2t = 0 \to x_1 = 2t - 2s$

$x_2 - t = 0 \to x_2 = t$

$x_5 = 0$

Solutions: $\begin{bmatrix} 2t-2s \\ t \\ s \\ t \\ 0 \end{bmatrix}$

$= s \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$

So a basis for null(A) $= \left\{ \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \right\}$.

To find a basis for row(A) (= row(R)):

take non-zero rows of R [Because of the 0s before all the leading 1s, we can see this is a lin. independent set]

Here : $\{[1, 0, 2, -2, 0], [0, 1, 0, -1, 0], [0, 0, 0, 0, 1]\}$

To find a basis for col(A) :
we can read off dependence relations between columns of R:

$$R = \begin{bmatrix} 1 & 0 & 2 & -2 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$
$$\begin{matrix} \bar v_1 & \bar v_2 & \bar v_3 & \bar v_4 & \bar v_5 \end{matrix}$$

Pivot columns $(\bar v_1, \bar v_2, \bar v_5)$ are standard unitvectors.
Write other columns in terms of pivot columns:

$$\bar v_3 = 2 \bar v_1$$
$$\bar v_4 = -2\bar v_1 - \bar v_2$$

Get same dependencies between columns of A:

$$\bar c_3 = 2 \bar c_1$$
$$\bar c_4 = -2\bar c_1 - \bar c_2$$

so columns of A corresponding to pivot columns of R are a basis for col(A)

i.e. $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ -1 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 14 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ -1 \\ -4 \\ 3 \end{bmatrix} \right\}.$

<u>Note</u> in general
$\text{col}(A) \neq \text{col}(R)$!

Example from last time Find a basis for

$$V = \text{span}\left\{ \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-2\\3\\1\end{bmatrix}, \begin{bmatrix}-2\\0\\-2\\0\end{bmatrix}, \begin{bmatrix}-2\\-2\\0\\1\end{bmatrix} \right\}.$$

Solution With $A = \begin{bmatrix}1 & 1 & -2 & -2\\ 0 & -2 & 0 & -2\\ 1 & 3 & -2 & 0\\ 0 & 1 & 0 & 1\end{bmatrix}$, $\text{col}(A) = V$

So reduce A to RREF $\longrightarrow \begin{bmatrix}1 & 0 & -2 & -3\\ 0 & 1 & 0 & 1\\ 0 & 0 & 0 & 0\\ 0 & 0 & 0 & 0\end{bmatrix}$

Pivot columns are 1st & 2nd so basis for $V = \text{col}(A)$ is $\left\{ \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\-2\\3\\1\end{bmatrix} \right\}$.

1st & 2nd columns of A.

So to summarize: given A, to find bases for null(A), col(A), row(A):

① Reduce A to its RREF R

② (a) null(A) Solve $A\bar{x} = \bar{0}$ using $[R \mid 0]$; basis for sol^n space = basis for null(A)

(b) Non-zero rows of R

row(A) are a basis for row(A) (= row(R)).

(c) If indices of the pivot columns of R are

col(A) $i_1, \dots, i_\ell$ (e.g. 1st, 3rd, 10th)

then $\{\bar{c}_{i_1}, \dots, \bar{c}_{i_\ell}\}$ is a basis for col(A)

columns of A

here in this example

$(\{\bar{c}_1, \bar{c}_3, \bar{c}_{10}\})$

& linear relationships between columns of A are true between columns of R (& vice versa).

More formally: if $\bar{v}_r$ is a (non-pivot) column of R with

$$\bar{v}_r = k_{i_1}\bar{v}_{i_1} + \dots + k_{i_\ell}\bar{v}_{i_\ell}$$

, where $\bar{v}_{i_j}$ are the pivot columns,

then

$$\bar{c}_r = k_{i_1}\bar{c}_{i_1} + \dots + k_{i_\ell}\bar{c}_{i_\ell}.$$

same coefficients

corresponding columns