1B03 – LINEAR ALGEBRA 1 (C01) WS19 Lecture 4

Last time Matrices are useful!

- For systems of L.E.s, turning the augmented matrix into an RREF matrix tells us about solution(s)
- Gauss–Jordan Elimination : a procedure for turning any matrix into an RREF matrix using Elementary Row Operations

$$\left(\text{matrix} \xrightarrow[\text{Gaussian elimination}]{} \text{REF matrix} \longrightarrow \text{RREF matrix}\right)$$

How many solutions?

2 parallel lines $\begin{Bmatrix} x-2y=-10 \\ 2x-4y=6 \end{Bmatrix} \longrightarrow \begin{bmatrix} 1 & -2 & -10 \\ 0 & 0 & 1 \end{bmatrix}$ in REF

$$\longrightarrow \begin{bmatrix} 1 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$ RREF

If you get a line $[0\ 0 \ldots 0\ 1]$ in RREF (or REF)

says $0=1$

then no solutions (inconsistent).

If consistent & there are free variables, then there are ∞-many (parametric) solutions & the RREF matrix gives the parameterization.

If no free variables & consistent, then there is 1 unique solution & RREF lets you read it off.

1.3 Matrices

<u>Definition</u> An $m \times n$ matrix is a rectangular array of numbers with m rows and n columns

<u>dimensions</u>

<u>Examples</u> $\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$ is a 2×3 matrix.

$\begin{bmatrix} 0 & 5 & -3 & 7 \end{bmatrix}$ is a 1×4 matrix

(sometimes written $(0, 5, -3, 7)$)

a $1 \times n$ matrix is called a row matrix or row vector

$\begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$ is a 3×1 matrix

any $m \times 1$ matrix is called a column matrix or column vector

A matrix is <u>square</u> if $m=n$ e.g. $\begin{bmatrix} 2 & 5 \\ -\pi & 16 \end{bmatrix}$

is a 2×2 matrix

Matrices : capital letters

$A, B, I, \ldots$

Entries : corresponding small letters

a_{ij} = entry in row i, column j of matrix A

(sometimes write $A = [a_{ij}]_{i,j}$)

<u>Example</u> In $A = \begin{bmatrix} 3 & 5 & 2 \\ -4 & 1 & 6 \end{bmatrix}$, $a_{11} = 3$

$a_{21} = -4$

$a_{12} = 5$

In special cases of row & column vectors, drop unnecessary subscript.

Use bold little letters: in practice we write: $\underline{a}$ or $\overline{a}$ or $\vec{a}$

$\underline{a} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ has $a_1 = 2$, $a_2 = 1$, $a_3 = 0$

$\overline{x} = [x_1 \ \ x_2 \ \ x_3]$.

Two matrices A, B are equal ($A = B$) if they match entry-for-entry i.e. $a_{ij} = b_{ij}$ for every pair i, j (in particular dimensions of A and B are the same)

"Is $\begin{bmatrix} 2 & 1 \\ 3 & 6 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}$?" is meaningless!

<u>Transpose</u> $\quad A \quad m \times n$ matrix

A^T – transpose of A

– $n \times m$ matrix

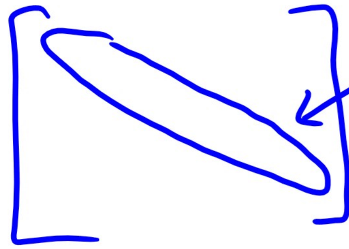
– interchange rows & columns of A

<u>Examples</u>

(1) $A = \begin{bmatrix} 1 & 3 \\ 5 & 7 \end{bmatrix} \qquad A^T = \begin{bmatrix} 1 & 5 \\ 3 & 7 \end{bmatrix}$

(2) $B = \begin{bmatrix} 1 \\ 0 \\ 3 \end{bmatrix} \qquad B^T = \begin{bmatrix} 1 & 0 & 3 \end{bmatrix}$

(3) $C = \begin{bmatrix} 1 & 5 & -1 \\ 1 & -8 & 2 \end{bmatrix} \qquad C^T = \begin{bmatrix} 1 & 1 \\ 5 & -8 \\ -1 & 2 \end{bmatrix}$

If A is square  main diagonal (NW-SE)

then trace of A $\operatorname{tr}(A)$ = sum of entries on main diagonal

Example $A = \begin{bmatrix} 3 & 5 & 7 \\ 2 & -1 & 4 \\ -7 & 6 & 5 \end{bmatrix}$ $\operatorname{tr}(A) = 3 - 1 + 5 = 7.$

$$\operatorname{tr}(A^T) = \operatorname{tr}\left(\begin{bmatrix} 3 & 2 & -7 \\ 5 & -1 & 6 \\ 7 & 4 & 5 \end{bmatrix}\right) = 7 = \operatorname{tr}(A).$$

(This always happens — if $\operatorname{tr}(A)$ defined (i.e. A square), then $\operatorname{tr}(A) = \operatorname{tr}(A^T)$!

Addition and Multiplications with Matrices

Addition - matrices must be same shape (same $m \times n$)

- add entries in matching positions

- Formally if A, B are $m \times n$ matrices, then $A + B = C$ where C is $m \times n$ matrix with $c_{ij} = a_{ij} + b_{ij}$ for all i, j

Example $A = \begin{bmatrix} 1 & -3 & 5 \\ -2 & 0 & 6 \end{bmatrix}$ $B = \begin{bmatrix} 0 & 1 & -4 \\ -1 & 2 & 7 \end{bmatrix}$

$$A + B = \begin{bmatrix} 1+0 & -3+1 & 5-4 \\ -2-1 & 0+2 & 6+7 \end{bmatrix} = \begin{bmatrix} 1 & -2 & 1 \\ -3 & 2 & 13 \end{bmatrix}.$$

Scalar Multiplication

- multiply all entries of a matrix by the same scalar (number)
- Formally kA has entries ka_{ij}

Example $A = \begin{bmatrix} -2 & 7 \\ 5 & 1 \end{bmatrix}$ $k = 3$: $3A = \begin{bmatrix} 3(-2) & 3(7) \\ 3(5) & 3(1) \end{bmatrix}$

$$= \begin{bmatrix} -6 & 21 \\ 15 & 3 \end{bmatrix}.$$

Matrix Multiplication

- do NOT just multiply corresponding entries!!

Let A be $m \times n$ matrix

Let B be $\underbrace{k}_{=n} \times l$ matrix

Product $C = AB$ ONLY DEFINED when $\overset{\#\text{columns of } A}{n} = \overset{\#\text{rows of } B}{k}$

Then C is $m \times l$ matrix.

To find entry c_{ij} of $C = AB$:

- take the ith <u>row</u> of A $\begin{bmatrix} a_{i1} & a_{i2} & \dots & a_{in} \end{bmatrix}$

& the jth <u>column</u> of B $\begin{bmatrix} b_{1j} \\ b_{2j} \\ \vdots \\ b_{nj} \end{bmatrix}$ $n = k$

& multiply matching entries & add up:

$$c_{ij} = a_{i1} b_{1j} + a_{i2} b_{2j} + \dots + a_{in} b_{nj}.$$

<u>Example</u> $A = \begin{bmatrix} 0 & 3 \\ -1 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 0 & 4 \end{bmatrix}$

$\overset{m}{2} \times \overset{n}{2}$ $\qquad$ $\overset{k}{2} \times \overset{l}{3}$

T.B.C.

AB defined

BA not defined.

$B : \overset{m}{2} \times \overset{n}{3}$ $\quad$ $A : \overset{k}{2} \times \overset{l}{2}$.

$m = 3 \neq 2 = k$.