1B03 – LINEAR ALGEBRA 1 (C01) WS19 — Lecture 5

Last time: Matrix Operations

- **Addition**: $C = A + B$ $\quad c_{ij} = a_{ij} + b_{ij}$ $\quad A, B, C$: same dimensions
- **Scalar Multiplication**: $C = kA$ $\quad c_{ij} = k a_{ij}$
- **Matrix Multiplication**: $C = AB$ $\quad A$ is $m \times n$ (# columns of A), B is $n \times l$ (# rows of B)

$$c_{ij} = a_{i1} b_{1j} + \dots + a_{in} b_{nj}$$

- ith row of A: $[a_{i1} \cdots a_{in}]$
- jth column of B: $\begin{bmatrix} b_{1j} \\ \vdots \\ b_{nj} \end{bmatrix}$

Example $\quad A = \begin{bmatrix} 0 & 3 \\ -1 & 2 \end{bmatrix}$ (2×2) $\qquad B = \begin{bmatrix} 1 & -2 & 0 \\ -1 & 0 & 4 \end{bmatrix}$ (2×3)

$$C = AB = \begin{bmatrix} -3 & 0 & 12 \\ -3 & 2 & 8 \end{bmatrix}$$

$c_{11} = 0.(1) + 3(-1) = -3 \qquad c_{12} = 0.(-2) + 3.(0) = 0$

$c_{13} = 0.0 + 3(4) = 12$

$c_{21} = (-1)(1) + 2(-1) = -3 \qquad c_{22} = (-1)(-2) + 2(0) = 2$

$c_{23} = (-1)(0) + 2(4) = 8$

Example $D = \begin{bmatrix} -3 & 2 & -1 \end{bmatrix}$ $\quad E = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}$

$1 \times 3 \qquad 3 \times 1$

$$DE = \begin{bmatrix} -3(2) + 2(-1) + (-1)(0) \end{bmatrix} = \begin{bmatrix} -8 \end{bmatrix}$$

$$F = \underset{3\times 1}{E}\,\underset{1\times 3}{D} = \begin{bmatrix} -6 & 4 & -2 \\ 3 & -2 & 1 \\ 0 & 0 & 0 \end{bmatrix} \qquad ED = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \begin{bmatrix} -3 & 2 & -1 \end{bmatrix}$$

$f_{11} = 2(-3) = -6 \quad f_{12} = 2(2) = 4 \quad f_{13} = 2(-1) = -2$

$f_{21} = (-1)(-3) = 3$

Notice DE is the dot product of D & E.

In general in $C = AB$, c_{ij} is the dot product of ith row of A & jth column of B.

Other ways we think about matrix multiplication:

Example
$$\underbrace{\begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 0 & 5 \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} -1 & 2 \\ 0 & 3 \end{bmatrix}}_{B} = \begin{bmatrix} -2 & -5 \\ -1 & 2 \\ 0 & 15 \end{bmatrix}$$

Columns of B $\bar{b}_1 = \begin{bmatrix} -1 \\ 0 \end{bmatrix}$ $\bar{b}_2 = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$

vectors not entries

$$A\bar{b}_1 = \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix}, \quad A\bar{b}_2 = \begin{bmatrix} -5 \\ 2 \\ 15 \end{bmatrix}.$$

<u>Fact</u> If AB defined, B has columns $[\bar{b}_1\, \bar{b}_2 \ldots \bar{b}_\ell]$ then AB is the matrix $[A\bar{b}_1\; A\bar{b}_2\; \ldots\; A\bar{b}_\ell]$

columns of AB

In above example
$$A\bar{b}_2 = \begin{bmatrix} 2 & -3 \\ 1 & 0 \\ 0 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} -5 \\ 2 \\ 15 \end{bmatrix}$$

$$= 2 \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} -3 \\ 0 \\ 5 \end{bmatrix}$$

<u>Fact</u>

When multiplying a matrix A by a column vector, the product is a <u>linear combination</u> (weighted sum) of columns of A:

If $A = [A_1\ A_2\ \ldots\ A_n]$ and $\bar{b} = \begin{bmatrix} b_1 \\ \vdots \\ b_n \end{bmatrix}$ ← #s not rows

columns of A

then $A\bar{b} = b_1 A_1 + b_2 A_2 + \ldots + b_n A_n$

scalar matrix

1.4 Properties of Matrix Operations

(see p. 39 of text for longer list)

- $A + B = B + A$ (if $A+B$, $B+A$ defined)
- $(A+B)+C = A+(B+C)$
- $k(A+B) = kA + kB$, k scalar
- $C(A+B) = CA + CB$ (Exercise: think dot products)
- $A(BC) = (AB)C$

Example $\underbrace{\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix}}_{A} \left(\underbrace{\begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix}}_{B} \underbrace{\begin{bmatrix} 2 \\ 1 \end{bmatrix}}_{C} \right)$

$$= \begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ -5 \end{bmatrix}$$

$$\left(\begin{bmatrix} 1 & 2 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 3 & -1 \\ 1 & 0 \end{bmatrix} \right) \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} 5 & -1 \\ -3 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

$$= \begin{bmatrix} 9 \\ -5 \end{bmatrix} =$$

<u>BUT</u>

- Sometimes $AB \neq BA$!!!

  Example above $ED \neq DE$.

<u>Example</u> $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} = 0$

$\neq$

$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

zero matrix

$A.0 = 0$

where this makes sense

Also shows:

- Sometimes $AB = 0$ but also $A \neq 0$, $B \neq 0$.
- Sometimes $AB = AC$ but $B \neq C$

$$\begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ -1 & -2 \end{bmatrix} = \begin{bmatrix} -2 & -4 \\ 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 2 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} 7 & 12 \\ -1 & -2 \end{bmatrix} //$$

Identities & Inverses

A matrix like

$$\begin{bmatrix} 1 & 0 & \cdots & & \\ & 1 & \ddots & 0 & 0 \\ 0 & & \ddots & & 0 \\ & 0 & & 1 & \\ 0 & 0 & & \ddots & 1 \end{bmatrix}$$

square

0s everywhere except main diagonal

main diagonal all 1s

is called an <u>identity matrix</u>.

If $n \times n$, written I_n. (Or just I if n obvious from context.)

<u>Example</u>

$$\begin{bmatrix} 3 & -1 & \pi \\ 7 & e & 15 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \leftarrow I_3$$

$$= \begin{bmatrix} 3 & -1 & \pi \\ 7 & e & 15 \end{bmatrix}$$

Also

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 & -1 & \pi \\ 7 & e & 15 \end{bmatrix} = \begin{bmatrix} 3 & -1 & \pi \\ 7 & e & 15 \end{bmatrix}$$

In general, if A is $m \times n$

then $I_m A = A I_n = A$.

In other words, identity matrices I_n are like 1 in the real numbers.