1B03 – LINEAR ALGEBRA 1 (C01) WS19 Lecture 7

Last time ELEMENTARY MATRICES

- Any matrix E you get from any I by applying an Elementary Row Operation

FACT If $I_m \xrightarrow{\text{E.R.O.}} E$ ← elementary

$m \times n \to A \xrightarrow[\text{E.R.O.}]{\text{same}} EA$

Example $I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 5R_1} \underbrace{\begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{E = \text{elementary}}$

$A = \begin{bmatrix} 1 & \pi & 6 \\ -e & 2 & 7 \\ 0 & 8 & 9 \end{bmatrix} \xrightarrow[\substack{\text{E.R.O.} \\ R_2 \to R_2 + 5R_1}]{\text{same}} \begin{bmatrix} 1 & \pi & 6 \\ -e+5 & 2+5\pi & 37 \\ 0 & 8 & 9 \end{bmatrix}$

$\|$

$EA = \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & \pi & 6 \\ -e & 2 & 7 \\ 0 & 8 & 9 \end{bmatrix} = \begin{bmatrix} 1 & \pi & 6 \\ 5-e & 5\pi+2 & 37 \\ 0 & 8 & 9 \end{bmatrix}$

Fact Elementary matrices E are always invertible.

Notice $I_m \xrightarrow{\text{E.R.O.}} E$

$E^{-1} \xrightarrow[\text{E.R.O.}]{\text{same}} EE^{-1} = I_m$

(Notice E is $m \times m$ square, so E^{-1} is $m \times m$ square.)

reverse

E.R.O. ← also an E.R.O.

So to get E^{-1} from I_m do reverse E.R.O. (see last lecture) from the E.R.O. used to get E. from I_m

In particular, E^{-1} is also an elementary matrix.

Algorithm for finding A^{-1}

line optional

1. If A is $n \times n$, write $[A \mid I_n]$

2. Take A to RREF using E.R.O.s and do the same E.R.O.s to I_n.

   We get $[R \mid B]$ where R is the RREF of A.

3. Conclude: If $R = I_n$, then $B = A^{-1}$!

   If $R \neq I_n$, then A is singular (A^{-1} does not exist)

Why does this work?

Suppose the E.R.O.s we did in step 2 to get R from A have elementary matrices:

Elementary matrix for 1st E.R.O. / El. matrix for 2nd E.R.O. / El. matrix for kth E.R.O.

$$E_1,\ E_2,\ \dots,\ E_k \quad (k \text{ steps}).$$

We have

$$[\,A \mid I\,]$$

$$\xrightarrow{\text{1st E.R.O.}} \quad [\,E_1 A \mid E_1\,] \qquad E_1 = E_1 I$$

$$\xrightarrow{\text{2nd E.R.O.}} \quad [\,E_2 E_1 A \mid E_2 E_1\,]$$

$$\longrightarrow \quad \vdots$$

$$\xrightarrow{\text{kth E.R.O.}} \quad \Big[\,\overbrace{\underbrace{E_k E_{k-1} \cdots E_2 E_1}_{B} A}^{R} \;\Big|\; \overbrace{E_k E_{k-1} \cdots E_1}^{B}\,\Big]$$

So $R = BA$.

If $R = I$, then $BA = I$ so $B = A^{-1}$.

If $R \neq I$, then R has a row of zeros.

(So R not invertible & by this, A cannot be invertible.)

<u>Notice</u> $R = E_k E_{k-1} \cdots E_2 E_1 A$

So if $R = I$, $E_k E_{k-1} \cdots E_2 E_1 A = I$

$$\Rightarrow \cancel{(E_k^{-1} E_k)} E_{k-1} \cdots E_2 E_1 A = E_k^{-1}$$

$$\Rightarrow \cancel{E_{k-1}^{-1} E_{k-1}} \cdots E_2 E_1 A = E_{k-1}^{-1} E_k^{-1}$$

$\vdots$

$$\Rightarrow A = E_1^{-1} E_2^{-1} \cdots E_k^{-1}$$

$E_1^{-1}, E_2^{-1}, \ldots, E_k^{-1}$

i.e. A is a product of elementary matrices

↳ remember, from above, E elementary $\Rightarrow$ E^{-1} elementary

(& if A is a product of a bunch of elementary matrices we can reverse all this to find $B = A^{-1}$).

<u>Example</u> $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$. Find A^{-1}.

<u>Solution</u> Write $\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 & 0 & 1 \end{array}\right]$

$R_3 \to R_3 - 2R_1$ $\qquad$ $E_1 \times$

$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & -1 & 2 & 0 & 1 & 0 \\ 0 & 1 & -5 & -2 & 0 & 1 \end{array}\right]$, $\quad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -2 & 1 & 0 \end{bmatrix}$

"typo"

$R_2 \to -R_2$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 1 & -5 & -2 & 0 & 1 \end{array}\right]$$

$E_2 \times$

$$E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$R_3 \to R_3 - R_2$

$E_3 \times$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & -3 & -2 & 1 & 1 \end{array}\right]$$

$$E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$$

$R_3 \to -\frac{1}{3} R_3$

$E_4 \times$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 3 & 1 & 0 & 0 \\ 0 & 1 & -2 & 0 & -1 & 0 \\ 0 & 0 & 1 & 2/3 & -1/3 & -1/3 \end{array}\right]$$

$$E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1/3 \end{bmatrix}$$

$R_1 \to R_1 - 3R_3$

$R_2 \to R_2 + 2R_3$

$\left.\begin{array}{l} E_5 \times \\ E_6 \times \end{array}\right\} E_6 E_5 \times$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & 1 \\ 0 & 1 & 0 & 4/3 & -5/3 & -2/3 \\ 0 & 0 & 1 & 2/3 & -1/3 & -1/3 \end{array}\right].$$

$$E_5 = \begin{bmatrix} 1 & 0 & -3 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

I_3 $\qquad$ $\| \; A^{-1}$

$$E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

$= E_6 E_5 E_4 E_3 E_2 E_1.$

So $A = E_1^{-1} E_2^{-1} E_3^{-1} E_4^{-1} E_5^{-1} E_6^{-1}$