1B03 – LINEAR ALGEBRA 1 (C01) WS19 Lecture 8

Last time Using Gauss–Jordan Elimination to find INVERSES:

1. If A is $n \times n$, write $[A \mid I_n]$.
2. Take A to RREF R using Elementary Row Operations $\longrightarrow [R \mid B]$
3. If $R = I_n$, then $B = A^{-1}$!
   If $R \neq I_n$, then A not invertible.

We saw last time using this that $A = \begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 1 \end{bmatrix}$

has $A^{-1} = \begin{bmatrix} -1 & 1 & 1 \\ 4/3 & -5/3 & -2/3 \\ 2/3 & -1/3 & -1/3 \end{bmatrix}$.

1.6 More on Linear Systems & Invertible Matrices

We can write any system of L.E.s using products of matrices:

$$\begin{bmatrix} a_{11}x_1 + & \cdots & + a_{1n}x_n \\ a_{21}x_1 + & \cdots & + a_{2n}x_n \\ \vdots & \ddots & \vdots \\ a_{m1}x_1 + & \cdots & + a_{mn}x_n \end{bmatrix} \begin{matrix} = \\ = \\ \vdots \\ = \end{matrix} \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}$$

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}}_{\bar{x}} = \underbrace{\begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}}_{\bar{b}}$$

$m \times n$ $n \times 1$ $m \times 1$

$$A \cdot \bar{x} = \bar{b}$$

From this we can find a solution to the system if A is invertible (includes $m = n$)

i.e. if we ever say "A is invertible" then you know for free that A must be square

$$\longrightarrow \quad \cancel{A^{-1}(A}\bar{x}) = A^{-1}\bar{b}$$

$$\bar{x} = A^{-1}\bar{b}$$

__Example__ Solve

$$\begin{aligned} x \quad\quad + 3z &= 10 \\ -y + 2z &= 2 \\ 2x + y + z &= -3 \end{aligned}$$

__Solution__ In matrix form:

$$\underbrace{\begin{bmatrix} 1 & 0 & 3 \\ 0 & -1 & 2 \\ 2 & 1 & 1 \end{bmatrix}}_{A} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 10 \\ 2 \\ -3 \end{bmatrix}$$

So

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = A^{-1} \begin{bmatrix} 10 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} -11 \\ 12 \\ 7 \end{bmatrix}$$

(For A^{-1} see above.) (Check!)

So $x = -11$, $y = 12$, $z = 7$.

The old way: turn system into

$$\begin{bmatrix} 1 & 0 & -3 & 10 \\ 0 & -1 & 2 & 2 \\ 2 & 1 & 1 & -3 \end{bmatrix}$$ ← augmented matrix

& perform EROs to get to RREF.

These are exactly the same EROs as we used to find A^{-1} (earlier example)

get
$$\begin{bmatrix} 1 & 0 & 0 & -11 \\ 0 & 1 & 0 & 12 \\ 0 & 0 & 1 & 7 \end{bmatrix}$$

Applying those EROs to $\begin{bmatrix} 10 \\ 2 \\ -3 \end{bmatrix}$ is the same as the matrix product $\underbrace{E_6 E_5 E_4 E_3 E_2 E_1}_{= A^{-1}} \begin{bmatrix} 10 \\ 2 \\ -3 \end{bmatrix}$

(For exact values of E_is see last time.)

A invertible means $A\bar{x} = \bar{b}$ has at least one solution ($\bar{x} = A^{-1}\bar{b}$). But is this the only solution? (Well, yes, it's an equation, & so the value of $\bar{x}$ must be $A^{-1}\bar{b}$.)

<u>Theorem</u> (some of this we've already seen.)

Let A be an $n \times n$ matrix.

The following statements are "equivalent":

(all true about A or all false about A)

(1) A is invertible

(2) The only solution to $A\bar{x} = \bar{0}$ is $\bar{x} = \bar{0}$.
(we call this the "trivial" solution).

(3) RREF of A is I.

(4) A is a product of elementary matrices.

(5) $A\bar{x} = \bar{b}$ is consistent for all $\bar{b}$ (i.e. ≥ 1 solution)

(6) $A\bar{x} = \bar{b}$ has <u>exactly one</u> solution for all $\bar{b}$.

<u>Example</u> Solve

$$\begin{aligned} x + y \quad\quad &= 3 \\ x + 2y + 2z &= -1 \\ -x \quad\quad + 2z &= -7 \end{aligned} \quad \text{AND} \quad \begin{aligned} x + y \quad\quad &= 2 \\ x + 2y + 2z &= 7 \\ -x \quad\quad + 2z &= 8 \end{aligned}$$

2 systems $A\bar{x} = \bar{b}_1 = \begin{bmatrix} 3 \\ -1 \\ 7 \end{bmatrix}$ AND $A\bar{x} = \bar{b}_2 = \begin{bmatrix} 2 \\ 7 \\ 8 \end{bmatrix}$

Solve both systems at same time by modifying our original method:

$$\left[\begin{array}{ccc|c|c} 1 & 1 & 0 & 3 & 2 \\ 1 & 2 & 2 & -1 & 7 \\ -1 & 0 & 2 & -7 & 8 \end{array}\right]$$

Reduce to RREF.

$\downarrow$ 2 EROs

$$\left[\begin{array}{ccc|c|c} 1 & 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 1 & 2 & -4 & 10 \end{array}\right]$$

$\downarrow$ 1 ERO

$$\left[\begin{array}{ccc|c|c} 1 & 1 & 0 & 3 & 2 \\ 0 & 1 & 2 & -4 & 5 \\ 0 & 0 & 0 & 0 & 5 \end{array}\right]$$

$0 = 5 \longrightarrow$ 2nd system has no solution (inconsistent)

$\downarrow$ 2 EROs

$$\left[\begin{array}{ccc|c|c} 1 & 0 & -2 & 7 & 2 \\ 0 & 1 & 2 & -4 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{array}\right]$$

$\rightarrow$ From this can find ∞-many solutions to 1st system

z is free variable

We have $A\bar{x} = \bar{b}_1$ is consistent

but $A\bar{x} = \bar{b}_2$ is inconsistent

So (5) in Theorem fails so (1)-(6) all fail; in particular A is not invertible.

So when (for our example above) is $A\bar{x} = \bar{b}$ consistent?

entries, not vectors!

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 2 \\ -1 & 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

augmented matrix: $\left[\begin{array}{ccc|c} 1 & 1 & 0 & b_1 \\ 1 & 2 & 2 & b_2 \\ -1 & 0 & 2 & b_3 \end{array}\right]$

G-J Elim. to go to RREF

"Typo"

$$\left[\begin{array}{ccc|c} 1 & \cancel{0}\,1 & 0 & b_1 \\ 0 & 1 & 2 & b_2 - b_1 \\ 0 & 1 & 2 & b_3 + b_1 \end{array}\right]$$

$$\left[\begin{array}{ccc|c} 1 & \cancel{0}\,1 & 0 & b_1 \\ 0 & 1 & 2 & b_2 - b_1 \\ 0 & 0 & 0 & b_3 - b_2 + 2b_1 \end{array}\right]$$

$$0 = b_3 - b_2 + 2b_1$$

So $A\bar{x} = \bar{b}$ is consistent exactly when $b_2 = b_3 + 2b_1$.