1B03 – LINEAR ALGEBRA 1 (C01) WS19 Lecture 9

Yesterday Theorem about Invertibility:

The following statements are equivalent for any $n \times n$ A:

(1) A is invertible.
(2) The only solution to the system $A\bar{x}=\bar{0}$ is $\bar{x}=\bar{0}$.
(3) The RREF of A is I_n.
(4) A is a product of elementary matrices.
(5) For all $\bar{b}$, the system $A\bar{x}=\bar{b}$ is consistent.
(6) For all $\bar{b}$, the system $A\bar{x}=\bar{b}$ has exactly 1 solution.

We said before, to find A^{-1} (if it exists)

write $[A \mid I]$ & reduce A to its RREF:

$[R \mid B]$

If $R=I$, then $B=A^{-1}$!
If $R \neq I$, then A not invertible. At the time I said "R has a row of zeros".

So R not invertible ... but why then is A not invertible?

Remember $R = \underbrace{E_k E_{k-1} \dots E_1}_{\text{These are all invertible}} A$

We'll show <u>Theorem</u> If A, B are $n \times n$ and AB is invertible, then so are A and B.

Then <u>IF</u> A invertible, $R = E_k E_{k-1} \dots E_1 A$ would have to be invertible.

(So if R not invertible, then one of E_i's or A must not be invertible i.e. A must not be invertible.)

Why is Theorem true?

First show B invertible using (2) from the Big Theorem (about invertibility)

i.e. look at $B\bar{x} = \bar{0}$

$$A(B\bar{x}) = A\bar{0} = \bar{0}$$

i.e. $(AB)\bar{x} = \bar{0}$. AB invertible so by (2) in Big Theorem (applied to AB), $\bar{x} = \bar{0}$. → AB invertible so the only solution to $AB\bar{x} = \bar{0}$ is $\bar{x} = \bar{0}$.

So Big Theorem (2) talking about B is true (the only solution to $B\bar{x} = \bar{0}$ is $\bar{x} = \bar{0}$)

so (1) is true i.e. B is invertible.

For A set $A\bar{x} = \bar{0}$

so $A(BB^{-1})\bar{x} = \bar{0}$ (since B invertible) we can write B^{-1}

$$AB(B^{-1}\bar{x}) = \bar{0}$$

Since AB invertible, by Big Theorem (2) $B^{-1}\bar{x} = \bar{0}$ (Only solution to AB(something) $= \bar{0}$ is something $= \bar{0}$.)

Since B^{-1} invertible $\bar{x} = \bar{0}$ (only solution to $B^{-1}\bar{x} = \bar{0}$ is $\bar{x} = \bar{0}$) ✓

We showed that $A\bar{x} = \bar{0}$ forced $\bar{x} = \bar{0}$ so (2) true about A. → So A invertible.

Another Fact Let A, B $n \times n$. If $AB = I$, then A, B invertible ($\&$ $A^{-1} = B$, $B^{-1} = A$, $BA = I$).

Why? We'll show B^{-1} exists.

Let $B\bar{x} = \bar{0}$.

$$\underbrace{AB}_{I}\bar{x} = A\bar{0} = \bar{0}$$

↑ By assumption

$\Rightarrow \bar{x} = 0$.

So B invertible by (2) of Big Theorem.

$AB = I$ so $A\underbrace{BB^{-1}}_{I} = B^{-1}$ i.e. $A = B^{-1}$.

Matrix Polynomials Let $A = \begin{bmatrix} 5 & -1 \\ -2 & 3 \end{bmatrix}$.

We are able to compute $A^2 = \begin{bmatrix} 27 & -8 \\ -16 & 11 \end{bmatrix}$

and $8A = \begin{bmatrix} 40 & -8 \\ -16 & 24 \end{bmatrix}$. (for example)

So if $p(x) = x^2 - 8x + 13$, we can make sense of $p(A) = A^2 - 8A + 13I$

(Any constant c becomes cI.)

$$p(A) = \begin{bmatrix} 27 & -8 \\ -16 & 11 \end{bmatrix} - \begin{bmatrix} 40 & -8 \\ -16 & 24 \end{bmatrix} + \begin{bmatrix} 13 & 0 \\ 0 & 13 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

We say A is a root of $p(x)$ if $p(A) = 0$.

$$p(x) = x^2 - 8x + 13$$

$$A = \begin{bmatrix} 5 & -1 \\ -2 & 3 \end{bmatrix}$$

Notice $\operatorname{tr}(A) = 5 + 3 = 8$

$$\det(A) = 5(3) - (-1)(-2)$$
$$= 13$$

Fun Fact: if $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

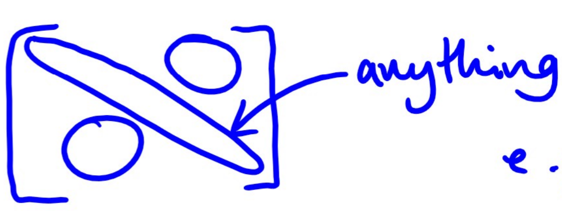
then A is a root of $x^2 - \operatorname{tr}(A)x + \underbrace{\det(A)}_{=0?}$

Question for later: how does this relate to inverses ...?

1.7 Special Kinds of Square Matrices:

Diagonal, Triangular, Symmetric

Diagonal $A = [a_{ij}]_{i,j}$ is diagonal if $a_{ij} = 0$ for $i \neq j$



e.g. $\begin{bmatrix} 5 & 0 \\ 0 & -2 \end{bmatrix}$, $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = I_3$, $\begin{bmatrix} 3 & 0 \\ 0 & 0 \end{bmatrix}$

If A, B are $n \times n$ diagonal, then $A \pm B$, kA are also diagonal.

k ← scalar

Also if A diagonal $n \times n$

B any $n \times n$ matrix, then

$$AB = \begin{bmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & & & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{bmatrix}$$

$$= \begin{bmatrix} a_{11}b_{11} & a_{11}b_{12} & \cdots & a_{11}b_{1n} \\ a_{22}b_{21} & a_{22}b_{22} & \cdots & a_{22}b_{2n} \\ \vdots & & & \\ a_{nn}b_{n1} & a_{nn}b_{n2} & \cdots & a_{nn}b_{nn} \end{bmatrix} = \left[a_{ii}b_{ij}\right]_{i,j}$$

i.e. scale all rows of B by corresponding entries of A.

Similarly $BA = \left[b_{ij}a_{jj}\right]_{i,j}$ (if A diagonal $n \times n$, B any $n \times n$)

i.e. scale all columns of B by corresponding entries of A.

e.g. $$\begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 5 \\ -6 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 15 \\ -6 & 1 \end{bmatrix}$$

$\times 3$

$\times 1$

$$\begin{bmatrix} 2 & 5 \\ -6 & 1 \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 6 & 5 \\ -18 & 1 \end{bmatrix}$$

$\times 3$

$\times 1$

If B happens to be diagonal $AB = BA$

$$= \begin{bmatrix} a_{11}b_{11} & & 0 \\ & \ddots & \\ 0 & & a_{nn}b_{nn} \end{bmatrix}$$

So Product of diagonal matrices is diagonal!

And if A is diagonal and invertible,

then $A^{-1} = \begin{bmatrix} 1/a_{11} & & & 0 \\ & 1/a_{22} & & \\ & & \ddots & \\ 0 & & & \frac{1}{a_{nn}} \end{bmatrix}$

A is diagonal & invertible exactly when

$a_{ii} \neq 0$ for all i

(If $a_{ii} \neq 0$ for all i, we can write down & clearly when you multiply it by A you get I. If $a_{ii} = 0$ for some i, then A has a row of zeros, so is not invertible.)

e.g. $\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$ ✓ invertible

$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ X not invertible

Also if A diagonal, $A^k = \begin{bmatrix} a_{11}^k & & 0 \\ & \ddots & \\ 0 & & a_{nn}^k \end{bmatrix}$

<u>Triangular</u> → Upper Triangular (U.T.) $a_{ij} = 0$ if $i > j$

→ Lower Triangular (L.T.)

$a_{ij} = 0$ if $i < j$

e.g. $\begin{bmatrix} 1 & 3 \\ 0 & 5 \end{bmatrix}$ ← U.T.

$\begin{bmatrix} 3 & 0 & 0 \\ -2 & 6 & 0 \\ 1 & -1 & 0 \end{bmatrix}$ ← L.T.

$\begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix}$ ← U.T., L.T. AND Diagonal

T. B. C.