MATH 1B03

I_n = $n \times n$ identity matrix

$$= \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ 0 & & 1 & \\ \vdots & & & \ddots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

$I_m \cdot A = A$, $A \cdot I_k = A$

A ← $m \times k$ matrix

If A is an $n \times n$ matrix, its inverse, if it exists, A^{-1}, is the unique $n \times n$ matrix with $A \cdot A^{-1} = I_n = A^{-1} \cdot A$

example $A = \begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix}$

$A^{-1} = \begin{bmatrix} -2 & -3 \\ -1 & -1 \end{bmatrix}$

$\begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} -2 & -3 \\ -1 & -1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

$\begin{bmatrix} -2 & -3 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -3 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

I_2

Properties of the inverse.

- If A is invertible, with inverse A^{-1} then A^{-1} is also invertible, and $(A^{-1})^{-1} = A$
- $A^n = \underbrace{A \cdot A \cdot A \cdots A}_{n\text{-times}}$. $(A^n)^{-1} = (A^{-1})^n$
- If $k \neq 0$, then kA is invertible, with $(kA)^{-1} = \frac{1}{k} \cdot A^{-1}$
- If A and B are $n \times n$, invertible matrices, then AB is invertible, and $(AB)^{-1} \overset{?}{=} \cancel{A^{-1} \cdot B^{-1}}$, $B^{-1}A^{-1}$

Proposed by a student in the class.

correct expression.

~~$(AB)(A^{-1}B^{-1}) = \leftarrow$ doesn't work~~

$(AB)(B^{-1}A^{-1}) = A(BB^{-1})A^{-1} = (A \cdot I_n) \cdot A^{-1} = A \cdot A^{-1} = I_n$

- A^T = the transpose of A
- If A is invertible, then $(A^T)^{-1} = (A^{-1})^T$

[Use $(AB)^T = B^T \cdot A^T$]

2 Reasons for a matrix to not be invertible ⇐ A is singular.

- If A has a row or column of just zeroes, then A is singular

eg $\underbrace{\begin{bmatrix} 1 & 3 & -1 \\ 0 & 0 & 0 \\ 0 & 5 & 17 \end{bmatrix}}_{A} \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} 1 & & \\ & \boxed{0} & \\ & & \end{bmatrix}$ always = 0

- If A has 2 rows or 2 columns that are identical, then A is singular.

eg $\underbrace{\begin{bmatrix} 1 & 2 & 7 \\ 0 & 5 & -3 \\ 1 & 2 & 7 \end{bmatrix}}_{A} \begin{bmatrix} \\ \\ \\ \end{bmatrix} = \begin{bmatrix} 1 & & \\ & & \\ 1 & & \end{bmatrix}$ ⇐ can't be the identity.

Solving linear systems using the inverse

- The 2x2 case: $\begin{matrix} ax+by=u \\ cx+dy=v \end{matrix} \iff \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_{= A} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$

$\iff A \cdot \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} u \\ v \end{bmatrix}$

Using A^{-1}, if it exists, we get:

$$A^{-1}\left(A\begin{bmatrix} x \\ y \end{bmatrix}\right) = A^{-1} \cdot \begin{bmatrix} u \\ v \end{bmatrix} \iff \underbrace{(\cancel{A^{-1}A})}_{I_2}\begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} u \\ v \end{bmatrix}$$

$$\iff \begin{bmatrix} x \\ y \end{bmatrix} = A^{-1}\begin{bmatrix} u \\ v \end{bmatrix}$$

Finding the inverse of a 2x2 matrix.

Let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Fact: A is invertible, if and only if $\underbrace{ad-bc}_{\text{the determinant of } A} \neq 0$

If $ad-bc \neq 0$, then

$$A^{-1} = \left(\frac{1}{ad-bc}\right)\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

Check: $A \cdot A^{-1} = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\left(\left(\frac{1}{ad-bc}\right)\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}\right)$

$A \cdot A^{-1}$

$$= \frac{1}{ad-bc}\begin{bmatrix} ad-bc & \cancelto{0}{a(-b)+ba} \\ \cancelto{0}{cd+d(-c)} & c(-b)+ad \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I_2$$

eg: If $A = \begin{bmatrix} 1 & -3 \\ -1 & 5 \end{bmatrix}$, then $A^{-1} = \begin{bmatrix} \frac{5}{2} & \frac{3}{2} \\ \frac{1}{2} & \frac{1}{2} \end{bmatrix}$

Section 1.5: Elementary Matrices and a method for Finding A^{-1}.

<u>Recall</u> the Elementary Row Operations

1. Scale a row by some $k \neq 0$ $\quad R_i \to k \cdot R_i$
2. Add a multiple of one row to another: $R_i \to R_i + k \cdot R_j$
3. Interchange two rows: $R_i \Leftrightarrow R_j$

Each ERO ~~has~~ can be reversed (or inverted)

1. $R_i \to \frac{1}{k} \cdot R_i$
2. $R_i \to R_i - kR_j$
3. $R_j \leftrightarrow R_i$

An elementary matrix is any matrix obtained from the identity matrix I_n by applying one E.R.O. to it.

---

eg: $I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 \to R_2 + 5R_1} \begin{bmatrix} 1 & 0 & 0 \\ 5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ an elementary matrix

$I_3 \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

Key Fact: Can encode ERO's on a matrix A by multiplying A, on the left, by the corresponding elementary matrix.

So $I_m \xrightarrow{\text{ERO}} E$ ⇐ elementary matrix

$\underset{\substack{\uparrow \\ m\times n}}{A} \xrightarrow{\substack{\text{same} \\ \text{ERO}}} EA$