Multiplying matrices by diagonal matrices is faster than usual matrix multiplication.

The following equations generalize to matrices of any size. Multiplying a matrix from the left by a diagonal matrix multiplies the rows by the diagonal entries:

$$\begin{bmatrix} \boxed{d_1} & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_1a_{12} & d_1a_{13} & d_1a_{14} \\ d_2a_{21} & d_2a_{22} & d_2a_{23} & d_2a_{24} \\ d_3a_{31} & d_3a_{32} & d_3a_{33} & d_3a_{34} \end{bmatrix}$$

Multiplying a matrix from the right by a diagonal matrix multiplies the columns by the diagonal entries:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} \\ a_{21} & a_{22} & a_{23} & a_{24} \\ a_{31} & a_{32} & a_{33} & a_{34} \end{bmatrix} \begin{bmatrix} d_1 & 0 & 0 & 0 \\ 0 & \boxed{d_2} & 0 & 0 \\ 0 & 0 & d_3 & 0 \\ 0 & 0 & 0 & d_4 \end{bmatrix} = \begin{bmatrix} d_1a_{11} & d_2a_{12} & d_3a_{13} & d_4a_{14} \\ d_1a_{21} & d_2a_{22} & d_3a_{23} & d_4a_{24} \\ d_1a_{31} & d_2a_{32} & d_3a_{33} & d_4a_{34} \end{bmatrix}$$

Question 7.9

Verify these equations via matrix multiplication.

Example 7.10

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} = \begin{bmatrix} 2a_{11} & 2a_{12} & 2a_{13} \\ a_{21} & a_{22} & a_{23} \\ 3a_{31} & 3a_{32} & 3a_{33} \end{bmatrix}$$

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix} = \begin{bmatrix} 2a_{11} & a_{12} & 3a_{13} \\ 2a_{21} & a_{22} & 3a_{23} \\ 2a_{31} & a_{32} & 3a_{33} \end{bmatrix}$$

Triangular matrices (defined only for square matrices)

The next best thing to a diagonal matrix is a triangular matrix.

Definition 7.11: Triangular matrices

A square matrix is upper triangular if the only nonzero entries are above or on the main diagonal.
A square matrix is lower triangular if the only nonzero entries are below or on the main diagonal.
A square matrix is triangular if it is either upper or lower triangular.

$$\underbrace{\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ 0 & a_{22} & a_{23} \\ 0 & 0 & a_{33} \end{bmatrix}}_{\text{upper}} \quad \underbrace{\begin{bmatrix} a_{11} & 0 & 0 \\ a_{21} & a_{22} & 0 \\ a_{31} & a_{32} & a_{33} \end{bmatrix}}_{\text{lower}}$$

Note that a diagonal matrix is both upper and lower triangular. A matrix in REF or RREF is upper triangular.

Fact 7.12: Properties of triangular matrices

1. The transpose of an upper triangular matrix is lower triangular, and vice versa.
2. The product of two upper triangular matrices is upper triangular. The product of two lower triangular matrices is lower triangular.
3. A triangular matrix is invertible if and only if its diagonal entries are non-zero.
4. The inverse of an upper triangular matrix is lower triangular. The inverse of a lower triangular matrix is upper triangular.

3. is very important and gives us a quick way to check if a triangular matrix is invertible. We will see deep reasons for these facts as we progress.

Example 7.13

$$\begin{bmatrix} 2 & 3 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 7 \end{bmatrix} \quad \begin{bmatrix} 4 & 0 & 0 \\ 5 & 4 & 0 \\ 1 & 12 & 1 \end{bmatrix}$$

<u>singular</u> <u>invertible</u>

Products of triangular matrices

If A and B are both upper triangular (or both lower triangular) matrices, we can easily determine the diagonal entries of the products AB and BA.

Fact 7.14: Products of triangular matrices

Let A and B be upper triangular matrices (or both lower triangular matrices). Then

$$(AB)_{ii} = (BA)_{ii} = (A)_{ii}\,(B)_{ii}$$

That is, the diagonal entries of AB and BA can be found by multiplying the diagonal entries of A and B. Remember that $AB \neq BA$, in general, so that the non-diagonal entries cannot be found this way.

Example 7.15

Let

$$A = \begin{bmatrix} 1 & 0 \\ 8 & 4 \end{bmatrix}, \; B = \begin{bmatrix} 2 & 0 \\ 5 & 3 \end{bmatrix}$$

They are both lower triangular, so the diagonal entries of AB and BA are

$$\begin{aligned} (AB)_{11} &= (BA)_{11} = 1 \cdot 2 = 2 \\ (AB)_{22} &= (BA)_{22} = 4 \cdot 3 = 12 \end{aligned}$$

But

$$AB = \begin{bmatrix} 1 & 0 \\ 8 & 4 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 5 & 3 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 36 & 12 \end{bmatrix}$$

and

$$BA = \begin{bmatrix} 2 & 0 \\ 5 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 8 & 4 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 29 & 12 \end{bmatrix}$$

Notice that the nondiagonal elements are not equal.

Symmetric matrices (defined only for square matrices)

Definition 7.16

A matrix A is symmetric if $A^T = A$.

Notice that only square matrices can be symmetric: if A is an $m \times n$ matrix, then A^T is an $n \times m$ matrix. If $A^T = A$ then the matrices must be of the same size, so that $m = n$ and A is square.

A diagonal matrix is automatically symmetric.

Example 7.17

The following matrices are symmetric

$$\begin{bmatrix} 4 & -2 & -2 \\ -2 & 3 & 0 \\ -2 & 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}, \quad \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

We can write the symmetric condition $A^T = A$ as

$$\begin{aligned} (A)_{ij} &= \left(A^T\right)_{ji}, \quad \text{by definition of the transpose} \\ &= (A)_{ji}, \quad \text{as } A^T = A \end{aligned}$$

Therefore A is symmetric if $(A)_{ij} = (A)_{ji}$.

Fact 7.18: Properties of symmetric matrices

Let A and B be symmetric matrices of the same size, and λ a scalar. Then

1. A^T is symmetric
2. $A + B$ and $A - B$ are symmetric
3. λA is symmetric

2. and 3. allow us to add, subtract, and multiply symmetric matrices by scalars to produce another symmetric matrix.

However, the product of two symmetric matrices is not necessarily symmetric. Consider the following example:

$$\begin{bmatrix} 3 & 4 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 8 & -3 \\ -3 & 0 \end{bmatrix} = \begin{bmatrix} 12 & \boxed{-9} \\ \boxed{29} & -12 \end{bmatrix}$$

as $29 \neq -9$, the resulting matrix is not symmetric.

Fact 7.19

If A is invertible and symmetric, then A^{-1} is symmetric also.

Proof: Let A be invertible and symmetric. Therefore A^{-1} exists and $A^T = A$. Recall that

$$(A^{-1})^T = (A^T)^{-1}.$$

Then

$$\begin{aligned} (A^{-1})^T &= (A^T)^{-1} \\ &= A^{-1} \end{aligned}$$

so A^{-1} is symmetric. ■

Let M be a matrix of any size. The products MM^T and M^TM are common in applications of matrix theory. These matrix products are always square.

Question 7.20

Check that MM^T and M^TM are square, where M is a matrix of any size.

Fact 7.21

Let M be a matrix of any size.

1. MM^T and M^TM are symmetric
2. If M is invertible, then MM^T and M^TM are invertible also

Example 7.22

Question: Find all constants a, b and c such that the matrix

$$A = \begin{bmatrix} 2 & a-2b+2c & 2a+b+c \\ 3 & 5 & a+c \\ 0 & -2 & 7 \end{bmatrix}$$

is symmetric.
Answer:

$$A^T = \begin{bmatrix} 2 & 3 & 0 \\ a-2b+2c & 5 & -2 \\ 2a+b+c & a+c & 7 \end{bmatrix}$$

If $A^T = A$ then

$$\begin{aligned} a-2b+2c &= 3 \\ 2a+b+c &= 0 \\ a+c &= -2 \end{aligned}$$

Solving this system yields

$$\begin{aligned} a &= -11 \\ b &= -9 \\ c &= -13 \end{aligned}$$

Question: Find a diagonal matrix D such that

$$D^5 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

Answer: If

$$D = \begin{bmatrix} d_1 & 0 & 0 \\ 0 & d_2 & 0 \\ 0 & 0 & d_3 \end{bmatrix}$$

then

$$D^5 = \begin{bmatrix} d_1^5 & 0 & 0 \\ 0 & d_2^5 & 0 \\ 0 & 0 & d_3^5 \end{bmatrix}$$

If

$$D^5 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}$$

then

$$\begin{aligned} d_1^5 &= -1, & d_2^5 &= 3, & d_3^5 &= 5 \\ d_1 &= -1, & d_2 &= 3^{\frac{1}{5}}, & d_3^5 &= 5^{\frac{1}{5}} \end{aligned}$$

so that

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 3^{\frac{1}{5}} & 0 \\ 0 & 0 & 5^{\frac{1}{5}} \end{bmatrix}$$

Suggested problems

Practice the material in this lecture by attempting the following problems in **Chapter 1.7** of Anton-Rorres, starting on page 72

- Questions 19, 21, 26, 34, 35, 41, 43, 44, 45
- True/False questions (b), (c), (f), (i)