MATH 1B03/1ZC3 Winter 2019

Lecture 8: Determinants I

Instructor: Dr Rushworth January 29th

Determinants via cofactor expansion

(from Chapter 2.1 of Anton-Rorres)

Matrices encode information. Often we don't need all of the information contained in a matrix, and wish to extract a certain part of it.

An example of this is trace of a matrix: given a square matrix A, the trace $tr(A)$ is a number containing some information about A.

A more important number we can extract from a matrix is the determinant. We have actually seen the determinant of a 2×2 matrix already: if

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

then the determinant of A is

$$det(A) = ad - bc$$

Recall that A is invertible if and only if $ad - bc \neq 0$. This result extends to square matrices of any size: a matrix is invertible if and only if it has non-zero determinant. For this and other reasons the determinant is an important quantity in linear algebra.

We already know how to compute the determinant of 2×2 matrices, and we will use this to compute the determinant of larger matrices. Given an $n \times n$ matrix, we compute its determinant in the following way:

1. break the matrix down into a collection of $(n-1) \times (n-1)$ matrices
2. break those down further into $(n-2) \times (n-2)$ matrices
3. keep breaking down until we produce a collection of 2×2 matrices

4. reassemble of the determinant of the $n \times n$ matrix from the collection of 2×2 determinants

(This is an example of mathematical induction.)

Definition 8.1: Determinant

Let A be a square matrix. The determinant of A is written $det(A)$ or $|A|$. It is a number.

Definition 8.2: Minors and Cofactors

Let A be an $n \times n$ matrix. Denote by $A[i, j]$ the matrix formed from A by deleting the i-th row and the j-th column.
The ij-th minor of A is the number

$$M_{i,j} := det(A[i, j]).$$

The ij-th cofactor of A is the number

$$C_{i,j} := (-1)^{i+j} M_{i,j}.$$

Warning: do not confuse the minor $M_{i,j}$ with the notation $(M)_{ij}$ for entries of a matrix.

To find $A[i, j]$, cover the i-th row and j-th column, then write down the remaining matrix. For example, if

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

then $A[2, 3]$ is found by considering

$$A = \begin{bmatrix} 1 & 2 & \text{[redacted]} \\ \text{[redacted]} & & \\ 7 & 8 & \end{bmatrix}$$

and

$$A[2, 3] = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$$

Example 8.3

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}, \quad A[1,1] = \begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}, \quad A[2,3] = \begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}$$

$$\begin{aligned} M_{1,1} &= det(A[1,1]) \\ &= det\left(\begin{bmatrix} 5 & 6 \\ 8 & 9 \end{bmatrix}\right) \\ &= 45 - 48 \\ &= -3 \end{aligned} \qquad \begin{aligned} M_{2,3} &= det(A[2,3]) \\ &= det\left(\begin{bmatrix} 1 & 2 \\ 7 & 8 \end{bmatrix}\right) \\ &= 8 - 14 \\ &= -6 \end{aligned}$$

$$\begin{aligned} C_{1,1} &= (-1)^{1+1} M_{1,1} \\ &= (-1)^2(-3) \\ &= -3 \end{aligned} \qquad \begin{aligned} C_{2,3} &= (-1)^{2+3} M_{2,3} \\ &= (-1)^5(-6) \\ &= 6 \end{aligned}$$

Note that there are more minors and cofactors of A to compute.

$$B = \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 \end{bmatrix}, \quad B[4,1] = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

but we don't know how to compute $M_{4,1}$ yet.

Question 8.4

Compute the remaining minors and cofactors of A in the example above.

Using minors and cofactors we can compute the determinant of matrices larger than 3×3. We are going to compute the determinant of larger matrices by computing lots of 2×2 determinants.

Fact 8.5: Cofactor expansion

Let $A = [a_{ij}]$ be an $n \times n$ matrix, and $C_{i,j}$ its cofactors. Then $det(A)$ can be found via cofactor expansion along the i-th row

$$\begin{aligned} det(A) &= \sum_{k=1}^{n} a_{ik} C_{i,k} \\ &= a_{i1}C_{i,1} + a_{i2}C_{i,2} + \cdots + a_{in}C_{i,n} \end{aligned}$$

or via cofactor expansion along the j-th column

$$\begin{aligned} det(A) &= \sum_{k=1}^{n} a_{kj} C_{k,j} \\ &= a_{1j}C_{1,j} + a_{2j}C_{2,j} + \cdots + a_{nj}C_{n,j} \end{aligned}$$

The idea: as A is $n \times n$, the cofactors $C_{i,j}$ are determinants of $(n-1) \times (n-1)$ matrices. To compute these determinants, we apply cofactor expansion again, and obtain determinants of $(n-2) \times (n-2)$ matrices. We keep applying cofactor expansion until we hit 2×2 determinants, which we know how to compute!

Its important to note that it doesn't matter which row or column we expand along: we will always arrive at the same answer.

Example 8.6

In the 3×3 case the formula for expansion along the i-th row is

$$det(A) = a_{i1}C_{i,1} + a_{i2}C_{i,2} + a_{i3}C_{i,3}$$

If we expand along the first row (so that $i = 1$) this becomes

$$det(A) = a_{11}C_{1,1} + a_{12}C_{1,2} + a_{13}C_{1,3}$$

Lets use this to compute $det(A)$ for

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & 2 \\ 1 & 0 & 5 \end{bmatrix}$$

The formula becomes

$$\begin{aligned} det(A) &= 1(-1)^{1+1}\begin{vmatrix}1 & 2\\0 & 5\end{vmatrix} + 2(-1)^{1+2}\begin{vmatrix}0 & 2\\1 & 5\end{vmatrix} + 3(-1)^{1+3}\begin{vmatrix}0 & 1\\1 & 0\end{vmatrix} \\ &= (-1)^2 5 + 2(-1)^3(-2) + 3(-1)^4(-1) \\ &= 5 + 4 - 3 \\ &= 6 \end{aligned}$$

Lets compute the determinant again, expanding along the 1-st column:

$$\begin{aligned} det(A) &= 1(-1)^{1+1}\begin{vmatrix}1 & 2\\0 & 5\end{vmatrix} + 0(-1)^{1+2}\begin{vmatrix}2 & 3\\0 & 5\end{vmatrix} + 1(-1)^{1+3}\begin{vmatrix}2 & 3\\1 & 2\end{vmatrix} \\ &= 5 + 1 \\ &= 6 \end{aligned}$$

What about a 4×4? We have to keep expanding. Expanding along the first row:

$$det\left(\begin{bmatrix}1 & 0 & 1 & 0\\0 & 2 & 1 & -1\\1 & 0 & 2 & -1\\0 & 1 & -1 & 0\end{bmatrix}\right) = 1(-1)^{1+1}\begin{vmatrix}2 & 1 & -1\\0 & 2 & -1\\1 & -1 & 0\end{vmatrix} + 0(-1)^{1+2}\begin{vmatrix}0 & 1 & -1\\1 & 2 & -1\\0 & -1 & 0\end{vmatrix}$$

$$+ 1(-1)^{1+3}\begin{vmatrix}0 & 2 & -1\\1 & 0 & -1\\0 & 1 & 0\end{vmatrix} + 0(-1)^{1+4}\begin{vmatrix}0 & 2 & 1\\1 & 0 & 2\\0 & 1 & -1\end{vmatrix}$$

to complete this determinant we need to repeat the process on the remaining 3×3 matrices.

The difficulty of computing matrix determinants grows very fast with the size of the matrix. In fact, the computation of the determinant of an $n \times n$ matrix requires $n! = n(n-1)(n-2)\cdots(2)(1)$ individual computations.

For example, a 5×5 determinant requires 120 calculations, and a 6×6 determinant requires 720 calculations!

This is an example of a task in linear algebra very well suited to computers, but not