1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

(WS 19 C03) Lecture 11

Yesterday — Determinants by Cofactor Expansion

$$\begin{bmatrix} a_{11} & a_{12} & \cdots & a_{1j} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2j} & \cdots & a_{2n} \\ \vdots & & & & & \\ a_{i1} & a_{i2} & \cdots & a_{ij} & \cdots & a_{in} \\ \vdots & & & & & \\ a_{n1} & a_{n2} & \cdots & a_{nj} & \cdots & a_{nn} \end{bmatrix}$$

- A — $n \times n$ matrix.
- Pick a row or column of A
- Go along the chosen row (or go down the chosen column) multiplying entries by their Cofactors: $C_{i,j} = (-1)^{i+j} \underbrace{M_{i,j}}$ ← Minor of a_{ij} $= \det(A[i,j])$.
- Add up the resulting products $= \det(A)$.

Useful trick for working out if $C_{i,j} = M_{i,j}$ or $C_{i,j} = -M_{i,j}$

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$$

Think chequerboard!

$$\begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix} \quad \text{etc.}$$

Useful insight: choose row or column to expand along/down that means least work! Look for rows/columns with lots of zeros!

Example $A = \begin{bmatrix} 3 & 1 & 0 & 5 \\ 6 & 7 & 2 & 3 \\ 0 & -2 & 0 & 2 \\ 0 & 0 & 0 & -1 \end{bmatrix}$. Find $\det(A)$.

Solution 3rd column has lots of zeros!
(so does 4th row!)

$$\det(A) = +0 - 2\begin{vmatrix} 3 & 1 & 5 \\ 0 & -2 & 2 \\ 0 & 0 & -1 \end{vmatrix} + 0 - 0$$

$$\begin{matrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{matrix}$$

$$= -2\left(3\begin{vmatrix} -2 & 2 \\ 0 & -1 \end{vmatrix}\right)$$

$$= -2\left(3\left((-2)(-1) - \cancelto{0}{2(0)}\right)\right)$$

$$= -12.$$

Notice $\begin{vmatrix} 3 & 1 & 5 \\ 0 & -2 & 2 \\ 0 & 0 & -1 \end{vmatrix} = 3(-2)(-1)$

The <u>determinant</u> of <u>any</u> Upper Triangular, Lower Triangular or Diagonal matrix (with diagonal $\begin{matrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{matrix}$)

is $\underline{\underline{a_{11} \cdot a_{22} \cdot \ldots \cdot a_{nn}}}$.

Remember A matrix A, U.T., L.T., or diagonal, $n \times n$ is invertible iff all diagonal entries $a_{ii} \neq 0$.

if and only if

Notice: This is the same as $\det(A) = a_{11} \cdot \ldots \cdot a_{nn} \neq 0$

(Already claimed — not yet justified — that $\det(A) \neq 0$ iff A invertible. This is consistent with that claim.)

Another useful trick for finding $\det(A)$ when A is 3×3 **ONLY** !!!

Example Let $A = \begin{bmatrix} 3 & 2 & -1 \\ 0 & -2 & 3 \\ -3 & 1 & 0 \end{bmatrix}$. Find $\det(A)$.

Solution Write out:

$$\begin{array}{ccccc} c1 & c2 & c3 & c1 & c2 \\ 3 & 2 & -1 & 3 & 2 \\ 0 & -2 & 3 & 0 & -2 \\ -3 & 1 & 0 & -3 & 1 \end{array}$$

$$\det(A) = 3(-2)(0) + 2(3)(-3) + (-1)(0)(1)$$
$$- (-1)(-2)(-3) - (3)(3)(1) - 2(0)(0)$$

$= 0 - 18 + 0 + 6 - 9 - 0 = -21.$

But wait! there's more!

2.2 Evaluating determinants by Row Reduction

Suppose $n \times n$ $A \xrightarrow{E.R.O.} B$.

What happens to $\det(A)$?

Depends on E.R.O.

E.R.O.		det (B)
(1) Scale a row by $k \neq 0$	$\rightarrow$	$\det(B) = k\det(A)$
(2) Add a row of A to another row of A	$\rightarrow$	$\det(B) = \det(A)$
(3) Swap over any two rows of A	$\rightarrow$	$\det(B) = -\det(A)$

In the special case that $A = I_n$ and $B = E$ elementary

$\det(I_n) = 1$

so in case (1) $\det(E) = k$
(2) $\det(E) = 1$
(3) $\det(E) = -1$

So E elem. $\Rightarrow \det(E) \neq 0$.

So for each type of ERO, notice that if

$$A \xrightarrow[\text{with elem. matrix } E]{\text{E.R.O.}} B = EA \quad \det(B) = \det(E) \times \det(A).$$

Notice if $\det(A) = 0$, det remains 0 under any ERO.