1ZC3 ENGINEERING MATH II-B (Linear Algebra I)

(WS 19 C03) Lecture 12

Last Time

- Deteminants of triangular & diagonal matrices are easy: $\det(A) = a_{11}a_{22}\cdots a_{nn}$.

- If $A \xrightarrow[\text{with elementary matrix } E]{\text{E.R.O.}} B = EA$, then $\det(B) = \det(E) \cdot \det(A)$,

$$\det(E) = \begin{cases} k & \text{if E.R.O. scales a row of } A \text{ by } k \neq 0 \\ 1 & \text{if E.R.O. adds a row of } A \text{ to another row of } A \\ -1 & \text{if E.R.O. swaps over 2 rows of } A. \end{cases}$$

$\neq 0$

Goal: Turn A into a triangular matrix with EROs

det. easy to find

Then recover det (A) by keeping track of EROs.

Gaussian elim. strategy will do the job (REF matrices are upper triangular)

but maybe we don't need to go all the way.

- Can stop at any stage you have a triangular matrix
- Also a row or column of zeros $\Rightarrow$ det. $= 0$.

Example Find $\begin{vmatrix} 1 & 5 & 3 \\ 2 & -1 & 2 \\ -1 & 0 & 6 \end{vmatrix}$ by Row Reduction.

Solution

$$= \underbrace{\begin{vmatrix} 1 & 5 & 3 \\ 0 & -11 & -4 \\ 0 & 5 & 9 \end{vmatrix}}_{A} = -11 \underbrace{\begin{vmatrix} 1 & 5 & 3 \\ 0 & 1 & 4/11 \\ 0 & 5 & 9 \end{vmatrix}}_{B}$$

$$\det(B) = -\frac{1}{11}\det(A)$$

$$\Rightarrow \det(A) = -11 \det(B)$$

$$= -11 \begin{vmatrix} 1 & 5 & 3 \\ 0 & 1 & 4/11 \\ 0 & 0 & \cancel{9 - \frac{20}{11}} \; \frac{79}{11} \end{vmatrix} = (-11)(1)(1)\left(\frac{79}{11}\right) = -79.$$

Example

Find $\begin{vmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & -2 & 0 \\ 1 & 0 & -1 & 2 \\ 2 & 1 & -2 & 4 \end{vmatrix}$ by Row Reduction.

Solution

$$= -\begin{vmatrix} 1 & 0 & -1 & 2 \\ -1 & 0 & -2 & 0 \\ 0 & 1 & 0 & 0 \\ 2 & 1 & -2 & 4 \end{vmatrix} = -\begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 0 & -3 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{vmatrix}$$

*

$$= \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & 1 & 0 & 0 \end{vmatrix} = \begin{vmatrix} 1 & 0 & -1 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -3 & 2 \\ 0 & 0 & 0 & 0 \end{vmatrix} = 0$$

* At this stage with 2 identical rows, we could abandon the Gaussian Elim. strategy, subtract one of the rows from the other & get a row of zeros $\Rightarrow \det = 0$.

Same idea if one row is multiple of another row.

There are also Elementary Column Operations

↳ exactly the same 3 types as EROs

↳ have analogous effect on determinant

e.g. swap 2 columns $\to$ multiply det. by -1

(add a column to another column $\to$ no change to det.)

(Scale a column by $k \neq 0 \to \det \times k$)

All this applies to columns as well as rows: look for columns of zeros & columns that are multiples of other columns.

Can mix & match operations if given free choice about how to find det.

Example

$$\begin{vmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 0 \\ 2 & 0 & -2 & 1 \\ 3 & -2 & 3 & -1 \end{vmatrix} = \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 2 & 0 & 0 & 1 \\ 3 & -2 & 6 & -1 \end{vmatrix} \; 5$$

Ideas in class: valid op., but output not Δ – needs more work

$$= \begin{vmatrix} 1 & 0 & 0 & -1/2 \\ 0 & -1 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & -2 & 6 & -5/2 \end{vmatrix}$$

(Column 4 $\to$ Column 4 $+ \frac{1}{2}$ Column 1)

OR $= -\begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 2 & 0 & 1 & 0 \\ 3 & 2 & -1 & 6 \end{vmatrix} = -1(-1)(1)(6) = 6.$

Swap Columns 3 & 4

$$= \begin{vmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 5 & -2 & 6 & 0 \\ 3 & -2 & 6 & -1 \end{vmatrix} = 1(-1)(6)(-1) = 6.$$

2.3 Properties of Determinants

Recall $A \xrightarrow[\text{elem. matrix } E]{\text{ERO}} B = EA \Rightarrow \det(B) = \det(E)\det(A)$

Also $\det(E) \neq 0$

Also: A invertible $\Leftrightarrow A = E_1 E_2 \cdot \ldots \cdot E_n$ some product of elem. matrices

So A invertible $\Leftrightarrow \det(A) = \underbrace{\det(E_1)}_{\neq 0} \cdot \ldots \cdot \underbrace{\det(E_n)}_{\neq 0}$

$\Rightarrow \det(A) \neq 0$

A not invertible $\Rightarrow$ RREF of A has a row of zeros

$\Rightarrow$ det of RREF is 0

$\Rightarrow \det(A) = 0.$

Also: for any 2 $n \times n$ matrices A and B

$\det(AB) = \det(A) . \det(B)$. (by similar reasoning)

In particular if A invertible

$$1 = \det(\underbrace{AA^{-1}}_{I}) = \det(A)\det(A^{-1})$$

So $\det(A^{-1}) = \dfrac{1}{\det(A)}$.

Let A be $n \times n$, $k \neq 0$

$$kA = \begin{bmatrix} k \times \text{Row } 1 \text{ of } A \\ k \times \text{Row } 2 \text{ of } A \\ \vdots \\ k \times \text{Row } n \text{ of } A \end{bmatrix}$$

So $\det(kA) = k^n \det(A)$.

__Bad News__ __Usually__ $\det(A+B) \neq \det(A) + \det(B)$.

__Example__ $A = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 0 & -1 \\ 1 & 2 \end{bmatrix}$

$$A+B = \begin{bmatrix} -1 & -1 \\ 1 & 3 \end{bmatrix}$$

$\det(A) = -1$

$\det(B) = -(-1) = 1$

$\det(A+B) = -3-(-1)$

$= -2$

$\neq -1+1$

Small compensation:

If A, B, C, all $n \times n$, all equal <u>except</u> for <u>one row</u> (call it Row i), and

Row i of C = Row i of A + Row i of B

Then $\det(C) = \det(A) + \det(B)$.

<u>Example</u> $A = \begin{bmatrix} 2 & 5 \\ 1 & 3 \end{bmatrix}$ $B = \begin{bmatrix} -1 & 6 \\ 1 & 3 \end{bmatrix}$

$$C = \begin{bmatrix} 1 & 11 \\ 1 & 3 \end{bmatrix}$$

$\det(A) = 6 - 5 = 1$ $\det(B) = -3 - 6 = -9$

$$\det(C) = 3 - 11 = -8 = 1 - 9$$

Can also substitute "column" for "row" everywhere in this fact.

<u>Example</u> $A = \begin{bmatrix} 2 & -1 & 0 \\ 1 & 0 & 5 \\ -3 & 2 & 2 \end{bmatrix}$ $B = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & -3 \\ -3 & 2 & -2 \end{bmatrix}$ $C = \begin{bmatrix} 2 & -1 & -1 \\ 1 & 0 & 2 \\ -3 & 2 & 0 \end{bmatrix}$

$\det(A) = 2(-10) - (-1)(2 - (-15))$
$= -20 + 17 = -3$
(along first row)

$\det(B) = (-1)(2 - (-2)) - (-3)(4 - 3)$
$= -4 + 3 = -1$
(along second row)

$\det(C) = (-1)(2) - 2(4 - 3)$
$= -2 - 2 = -4$
(down third column)
$= -3 - 1$

Hint: Use $\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix}$!